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# Integrable discretizations and self-adaptive moving mesh method for a coupled short pulse equation

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**Abstract**
In the present paper, integrable semi-discrete and fully discrete analogues of a coupled short pulse (CSP) equation are constructed. The key to the construction are the bilinear forms and determinant structure of the solutions of the CSP equation. We also construct $N$-soliton solutions for the semi-discrete and fully discrete analogues of the CSP equations in the form of Casorati determinants. In the continuous limit, we show that the fully discrete CSP equation converges to the semi-discrete CSP equation, then further to the continuous CSP equation. Moreover, the integrable semi-discretization of the CSP equation is used as a self-adaptive moving mesh method for numerical simulations. The numerical results agree with the analytical results very well.

Keywords: coupled short pulse equation, integrable discretization, self-adaptive moving mesh method

## 1. Introduction

The short pulse (SP) equation

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx} \tag{1.1}$$

was derived by Schäfer and Wayne to describe the propagation of ultra-short optical pulses in nonlinear media [1, 2]. Here, $u = u(x, t)$ represents the magnitude of the electric field, while the subscripts $t$ and $x$ denote partial differentiations. The SP equation represents an alternative approach in contrast with the slowly varying envelope approximation which leads to the nonlinear Schödinger (NLS) equation. As the pulse duration shortens, the NLS equation becomes less accurate, while the SP equation provides an increasingly better approximation to the corresponding solution of the Maxwell equations [2]. With the rapid progress of ultra-short optical pulse techniques, it is expected that the SP equation and its multi-component generalization will play more and more important roles in applications.

The SP equation has been shown to be completely integrable[3–5]. The loop-soliton solutions as well as smooth-soliton solutions of the SP solution were found in [6–8]. Multi-soliton solutions including multi-loop and multi-breather ones were given in [9]. Periodic solutions to the SP equation were discussed in [10].

Similar to the case of the NLS equation [11], it is necessary to consider its two-component or multi-component generalizations of the SP equation for describing the effects of polarization or anisotropy. As a matter of fact, several integrable coupled SP equations have been proposed in the literature [12–17]. A complex version of the integrable coupled SP equation in [14, 15] was studied in [18]. The bi-Hamiltonian structures for the above two-component SP equations were obtained in [19].

Integrable discretizations of soliton equations have received considerable attention recently [20–23]. Integrable semi- and full discretizations of the SP equation were constructed via Hirota's bilinear method [24]. The same discretizations were reconstructed from the point of view of geometry in [25]. Most recently, an integrable discretization for a coupled SP equation proposed in [14, 15] was constructed [26].

In the present paper, we consider integrable discretizations of another coupled short pulse (CSP) equation proposed by one of the authors [16]

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx} + \frac{1}{2}v^2u_{xx}, \tag{1.2}$$

$$v_{xt} = v + \frac{1}{6}\left(v^3\right)_{xx} + \frac{1}{2}u^2v_{xx}. \tag{1.3}$$

It was shown in [16] that equations (1.2) and (1.3) can be derived from bilinear equations

$$D_sD_yf\cdot f = \frac{1}{2}\left(f^2 - \bar{f}^2\right), \tag{1.4}$$

$$D_sD_y\bar{f}\cdot\bar{f} = \frac{1}{2}\left(\bar{f}^2 - f^2\right), \tag{1.5}$$

$$D_sD_yg\cdot g = \frac{1}{2}\left(g^2 - \bar{g}^2\right), \tag{1.6}$$

$$D_sD_y\bar{g}\cdot\bar{g} = \frac{1}{2}\left(\bar{g}^2 - g^2\right), \tag{1.7}$$

through a hodograph transformation

$$x = y - \left(\ln\left(F\bar{F}\right)\right)_s, \quad t = s, \tag{1.8}$$

and dependent variable transformations

$$u = \mathrm{i}\left(\ln \frac{\bar{F}}{F}\right)_s, \quad v = \mathrm{i}\left(\ln \frac{\bar{G}}{G}\right)_s, \tag{1.9}$$

where $F = fg$, $G = f\bar{g}$, $\bar{F}$ and $\bar{G}$ stand for the complex conjugates of $F$ and $G$, respectively. Meanwhile, $N$-soliton solutions of the CSP equation in parametric form are given, and the properties of one-soliton, soliton-breather solutions are investigated in detail in [16]. The bi-Hamiltonian structure of the CSP equations (1.2) and (1.3) was derived by Brunelli and Sakovich [19].

The rest of the present paper is organized as follows. In section 2, we propose an integrable semi-discrete analogue of the CSP equation by constructing a Bäcklund transform of the bilinear equations of the CSP equation. Meanwhile, an $N$-soliton solution is provided in terms of the Casorati determinant form. In section 3, starting from two sets of Bäcklund transforms to the bilinear equations of the CSP equation, the fully discrete analogue of the CSP equation is proposed by introducing two auxiliary variables. Moreover, the $N$-soliton solution is presented to confirm the integrability. Section 4 contributes to the self-adaptive moving mesh method by applying the semi-implicit Euler scheme to the semi-discrete CSP. The paper is concluded by section 5. Appendices A, B and C present the proofs of proposition 1, theorems 1 and 2, respectively.

## 2. Integrable semi-discretization of the CSP equation

We start with two sets of bilinear equations for the semi-discrete two-dimensional Toda-lattice (2DTL) equations with the same discrete parameter $a$

$$\left(\frac{1}{a}D_{x_{-1}} - 1\right)\tau_n(k+1)\cdot\tau_n(k) + \tau_{n+1}(k+1)\tau_{n-1}(k) = 0, \tag{2.1}$$

$$\left(\frac{1}{a}D_{x_{-1}} - 1\right)\tau'_n(k+1)\cdot\tau'_n(k) + \tau'_{n+1}(k+1)\tau'_{n-1}(k) = 0, \tag{2.2}$$

which is linked by a Bäcklund transformation [27]

$$\left(D_{x_{-1}} - 1\right)\tau_n(k)\cdot\tau'_n(k) + \tau_{n+1}(k)\tau'_{n-1}(k) = 0. \tag{2.3}$$

**Proposition 1.** *The bilinear equations (2.1) and (2.3) admit the following determinant solutions*

$$\tau_n(x_{-1}, k) = \begin{vmatrix} \phi_n^{(1)}(k) & \phi_{n+1}^{(1)}(k) & \cdots & \phi_{n+N-1}^{(1)}(k) \\ \phi_n^{(2)}(k) & \phi_{n+1}^{(2)}(k) & \cdots & \phi_{n+N-1}^{(2)}(k) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_n^{(N)}(k) & \phi_{n+1}^{(N)}(k) & \cdots & \phi_{n+N-1}^{(N)}(k) \end{vmatrix}, \tag{2.4}$$

$$\tau'_n(x_{-1}, k) = \begin{vmatrix} \psi_n^{(1)}(k) & \psi_{n+1}^{(1)}(k) & \cdots & \psi_{n+N-1}^{(1)}(k) \\ \psi_n^{(2)}(k) & \psi_{n+1}^{(2)}(k) & \cdots & \psi_{n+N-1}^{(2)}(k) \\ \cdots & \cdots & \cdots & \cdots \\ \psi_n^{(N)}(k) & \psi_{n+1}^{(N)}(k) & \cdots & \psi_{n+N-1}^{(N)}(k) \end{vmatrix}, \tag{2.5}$$

*where*

$$\phi_n^{(i)}(k) = p_i^n\left(1 - ap_i\right)^{-k}e^{\xi_i} + q_i^n\left(1 - aq_i\right)^{-k}e^{\eta_i}, \tag{2.6}$$

$$\psi_n^{(i)}(k) = p_i^n\left(1 - p_i\right)\left(1 - ap_i\right)^{-k}e^{\xi_i} + q_i^n\left(1 - q_i\right)\left(1 - aq_i\right)^{-k}e^{\eta_i}, \tag{2.7}$$

*with*

$$\xi_i = p_i^{-1}x_{-1} + \xi_{i0}, \quad \eta_i = q_i^{-1}x_{-1} + \eta_{i0}.$$

*Here $p_i$, $q_i$, $\xi_{i0}$ and $\eta_{i0}$ are arbitrary parameters which can take either real or complex values.*

The proof is presented in appendix A.

Applying a 2-reduction condition $q_i = -p_i$, then we could have each of the $\tau$ sequences become a sequence of period 2, i.e., $\tau_n \doteqdot \tau_{n+2}$, $\tau'_n \doteqdot \tau'_{n+2}$. Here $\doteqdot$ means two $\tau$ functions are equivalent up to a constant multiple. Furthermore, by choosing particular values of phase constants, we can make $\tau_n$ and $\tau_{n+1}$ complex conjugate to each other. Based on the bilinear equations with 2-reduction, we construct the semi-discrete analogue of the CSP equation using the following theorem:

**Theorem 1.** *The following equations constitute an integrable semi-discretization of the CSP equations (1.2) and (1.3)*

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(u_{k+1} - u_k\right) = \frac{1}{2}\delta_k\left(u_{k+1} + u_k\right) - \frac{1}{2\delta_k}\left(u_{k+1} - u_k\right)\left(v_{k+1}^2 - v_k^2\right), \tag{2.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(v_{k+1} - v_k\right) = \frac{1}{2}\delta_k\left(v_{k+1} + v_k\right) - \frac{1}{2\delta_k}\left(v_{k+1} - v_k\right)\left(u_{k+1}^2 - u_k^2\right), \tag{2.9}$$

$$\frac{\mathrm{d}\delta_k}{\mathrm{d}\,s} = -\frac{1}{2}\left(u_{k+1}^2 - u_k^2 + v_{k+1}^2 - v_k^2\right). \tag{2.10}$$

*Furthermore, the N-soliton solution to the above semi-discrete CSP equation is of the following form*

$$u_k = \mathrm{i}\ln\left(\frac{\bar{f}_k\,\bar{g}_k}{f_k\,g_k}\right)_s, \quad v_k = \mathrm{i}\ln\left(\frac{\bar{f}_k\,g_k}{f_k\,\bar{g}_k}\right)_s, \tag{2.11}$$

$$x_k = 2ka - \left(\ln\left(f_k\bar{f}_k\,g_k\bar{g}_k\right)\right)_s, \tag{2.12}$$

$$\delta_k = x_{k+1} - x_k = \frac{a}{2}\left(\frac{f_{k+1}f_k}{\bar{f}_{k+1}\bar{f}_k} + \frac{\bar{f}_{k+1}\bar{f}_k}{f_{k+1}f_k} + \frac{g_{k+1}g_k}{\bar{g}_{k+1}\bar{g}_k} + \frac{\bar{g}_{k+1}\bar{g}_k}{g_{k+1}g_k}\right), \tag{2.13}$$

*where $f_k$, $g_k$, $\bar{f}_k$ and $\bar{g}_k$ are tau-functions defined by*

$$f_k = \tau_0\left(\frac{s}{2}, k\right), \quad \bar{f}_k = \tau_1\left(\frac{s}{2}, k\right), \quad g_k = \tau'_0\left(\frac{s}{2}, k\right), \quad \bar{g}_k = \tau'_1\left(\frac{s}{2}, k\right), \tag{2.14}$$

*with*

$$\tau_n\left(\frac{s}{2}, k\right) = \left|\phi_{(n+j-1)}^{(i)}\right|_{1\leqslant i,j\leqslant N}, \quad \tau'_n\left(\frac{s}{2}, k\right) = \left|\psi_{(n+j-1)}^{(i)}\right|_{1\leqslant i,j\leqslant N}, \tag{2.15}$$

$$\phi_n^{(i)}(k) = p_i^n\left(1 - ap_i\right)^{-k} e^{\frac{1}{2p_i}s+\xi_{i0}} + (-p_i)^n\left(1 + ap_i\right)^{-k} e^{-\frac{1}{2p_i}s+\eta_{i0}},$$

$$\psi_n^{(i)}(k) = p_i^n\left(1 - p_i\right)\left(1 - ap_i\right)^{-k} e^{\frac{1}{2p_i}s+\xi_{i0}} + (-p_i)^n\left(1 + p_i\right)\left(1 + ap_i\right)^{-k} e^{-\frac{1}{2p_i}s+\eta_{i0}}.$$

The proof is presented in appendix B. In the process of the proof, the multi-soliton solution expressed in determinant form is obvious. Next, we show that the semi-discrete CSP equation converges to the CSP equation in the continuous limit.

In the continuous limit $a \to 0$ ($\delta_k \to 0$), we have

$$\frac{u_{k+1} - u_k}{\delta_k} \to \frac{\partial u}{\partial x}, \qquad \frac{u_{k+1} + u_k}{2} \to u, \tag{2.16}$$

$$\frac{\partial x}{\partial s} = \frac{\partial x_0}{\partial s} + \sum_{j=0}^{k-1}\frac{d\delta_j}{ds} = -\frac{1}{2}\sum_{j=0}^{k-1}\left(u_{j+1}^2 + v_{j+1}^2 - u_j^2 - v_j^2\right) \to -\frac{1}{2}\left(u^2 + v^2\right). \tag{2.17}$$

Thus

$$\partial_s = \partial_t + \frac{\partial x}{\partial s}\partial_x \to \partial_t - \frac{1}{2}\left(u^2 + v^2\right)\partial_x. \tag{2.18}$$

Consequently, equation (B.25) converges to

$$\left(\partial_t - \frac{1}{2}\left(u^2 + v^2\right)\partial_x\right)u_x = u\left(1 + u_x^2\right),$$

which is nothing but the first equation of the CSP equation (1.2).

It can be shown in the same way that equation (B.26) converges to equation (1.3), the second equation of the CSP equation.

## 3. Fully discretization of the CSP equation

To construct a fully discrete analogue of the CSP equation, we introduce one more discrete variable $l$ which corresponds to the discrete time variable.

It is known in [29] that the $\tau$-functions

$$\tau_n(k, l) = \left|\phi_{(n+j-1)}^{(i)}(k, l)\right|_{1\leqslant i,j\leqslant N}, \quad \tau'_n(k, l) = \left|\psi_{(n+j-1)}^{(i)}(k, l)\right|_{1\leqslant i,j\leqslant N}, \tag{3.1}$$

with

$$\phi_n^{(i)}(k, l) = p_i^n\left(1 - ap_i\right)^{-k}\left(1 - b\frac{1}{p_i}\right)^{-l} e^{\frac{1}{2p_i}s+\xi_{i0}} + q_i^n\left(1 - aq_i\right)^{-k}\left(1 - b\frac{1}{q_i}\right)^{-l} e^{\frac{1}{2q_i}s+\eta_{i0}},$$

$$\psi_n^{(i)}(k, l) = p_i^n\left(1 - p_i\right)\left(1 - ap_i\right)^{-k}\left(1 - b\frac{1}{p_i}\right)^{-l} e^{\frac{1}{2p_i}s+\xi_{i0}} + q_i^n\left(1 - q_i\right) \times \left(1 - aq_i\right)^{-k}\left(1 - b\frac{1}{q_i}\right)^{-l} e^{\frac{1}{2q_i}s+\eta_{i0}}$$

satisfy bilinear equations

$$\left(\frac{2}{a}D_s - 1\right)\tau_n(k+1, l) \cdot \tau_n(k, l) + \tau_{n+1}(k+1, l)\tau_{n-1}(k, l) = 0, \tag{3.2}$$

$$\left(\frac{2}{a}D_s - 1\right)\tau'_n(k+1, l) \cdot \tau'_n(k, l) + \tau'_{n+1}(k+1, l)\tau'_{n-1}(k, l) = 0, \tag{3.3}$$

and

$$(2bD_s - 1)\tau_n(k, l+1) \cdot \tau_{n+1}(k, l) + \tau_n(k, l)\tau_{n+1}(k, l+1) = 0. \tag{3.4}$$

$$(2bD_s - 1)\tau'_n(k, l+1) \cdot \tau'_{n+1}(k, l) + \tau'_n(k, l)\tau'_{n+1}(k, l+1) = 0. \tag{3.5}$$

Here $n$, $k$, $l$ are integers, $a$, $b$ are real numbers, $p_i$, $q_i$, $\xi_{i0}$, $\eta_{i0}$ are arbitrary complex numbers.

By applying a 2-reduction condition: $q_i = -p_i$, we have $\tau_n \doteqdot \tau_{n+2}$, $\tau'_n \doteqdot \tau'_{n+2}$. We can further get

$$f_{k,l} = \tau_0(k, l), \quad \bar{f}_{k,l} = \tau_1(k, l), \quad g_{k,l} = \tau'_0(k, l), \quad \bar{g}_{k,l} = \tau'_1(k, l),$$

by adjusting the phases in $\phi_i^{(n)}(k, l)$ and $\psi_i^{(n)}(k, l)$. Here $\bar{f}$ and $\bar{g}$ represent complex conjugate functions of $f$ and $g$, respectively. A fully discrete CSP equation can be constructed as follows:

**Theorem 2.** *The fully discrete analogue of the CSP equations (1.2) and (1.3) is of the form*

$$\begin{aligned}
&\frac{1}{b}(u_{k+1,l+1} - u_{k+1,l} - u_{k,l+1} + u_{k,l} + v_{k+1,l+1} - v_{k+1,l} - v_{k,l+1} + v_{k,l}) \\
&\quad = (y_{k+1,l} - y_{k,l+1})(u_{k+1,l+1} + u_{k,l} + v_{k+1,l+1} + v_{k,l}) \\
&\qquad + (y_{k+1,l+1} - y_{k,l})(u_{k,l+1} + u_{k+1,l} + v_{k,l+1} + v_{k+1,l}),
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&\frac{1}{b}(u_{k+1,l+1} - u_{k+1,l} - u_{k,l+1} + u_{k,l} - v_{k+1,l+1} + v_{k+1,l} + v_{k,l+1} - v_{k,l}) \\
&\quad = (z_{k+1,l} - z_{k,l+1})(u_{k+1,l+1} + u_{k,l} - v_{k+1,l+1} - v_{k,l}) \\
&\qquad + (z_{k+1,l+1} - z_{k,l})(u_{k,l+1} + u_{k+1,l} - v_{k,l+1} - v_{k+1,l}),
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&(y_{k+1,l+1} - y_{k+1,l} - y_{k,l+1} + y_{k,l})\left(\frac{1}{b} + y_{k,l+1} - y_{k+1,l}\right) \\
&\quad = -\frac{1}{4}(u_{k,l+1} + u_{k+1,l} + v_{k,l+1} + v_{k+1,l}) \\
&\qquad \times (u_{k+1,l+1} + u_{k+1,l} - u_{k,l+1} - u_{k,l} + v_{k+1,l+1} + v_{k+1,l} - v_{k,l+1} - v_{k,l}),
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&(z_{k+1,l+1} - z_{k+1,l} - z_{k,l+1} + z_{k,l})\left(\frac{1}{b} + z_{k,l+1} - z_{k+1,l}\right) \\
&\quad = -\frac{1}{4}(u_{k,l+1} + u_{k+1,l} - v_{k,l+1} - v_{k+1,l}) \\
&\qquad \times (u_{k+1,l+1} + u_{k+1,l} - u_{k,l+1} - u_{k,l} - v_{k+1,l+1} - v_{k+1,l} + v_{k,l+1} + v_{k,l}),
\end{aligned} \tag{3.9}$$

*where*

$$u_{k,l} = \mathrm{i}\ln\left(\frac{\bar{f}_{k,l}\,\bar{g}_{k,l}}{f_{k,l}\,g_{k,l}}\right)_s, \quad v_{k,l} = \mathrm{i}\ln\left(\frac{\bar{f}_{k,l}\,g_{k,l}}{f_{k,l}\,\bar{g}_{k,l}}\right)_s, \tag{3.10}$$

$$y_{k,l} = ka - \left(\ln\left(f_{k,l}\bar{f}_{k,l}\right)\right)_s, \quad z_{k,l} = ka - \left(\ln\left(g_{k,l}\bar{g}_{k,l}\right)\right)_s, \tag{3.11}$$

*and*

$$x_{k,l} = y_{k,l} + z_{k,l} = 2ka - \left(\ln\left(f_{k,l}\bar{f}_{k,l}g_{k,l}\bar{g}_{k,l}\right)\right)_s. \tag{3.12}$$

The proof is presented in appendix C.

Finally we show that equations (3.6)–(3.9) converge to the semi-discrete CSP equations (2.8)–(2.10) by taking a continuous limit in time ($b \to 0$). Under this limit, equations (3.6)–(3.9) become

$$\frac{\mathrm{d}}{\mathrm{d}s}(u_{k+1} - u_k) + \frac{\mathrm{d}}{\mathrm{d}s}(v_{k+1} - v_k) = (y_{k+1} - y_k)(u_{k+1} + u_k + v_{k+1} + v_k), \tag{3.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(u_{k+1} - u_k) - \frac{\mathrm{d}}{\mathrm{d}s}(v_{k+1} - v_k) = (z_{k+1} - z_k)(u_{k+1} + u_k - v_{k+1} - v_k), \tag{3.14}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(y_{k+1} - y_k) = -\frac{1}{4}(u_{k+1} + u_k + v_{k+1} + v_k)(u_{k+1} - u_k + v_{k+1} - v_k), \tag{3.15}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(z_{k+1} - z_k) = -\frac{1}{4}(u_{k+1} + u_k - v_{k+1} - v_k)(u_{k+1} - u_k - v_{k+1} + v_k), \tag{3.16}$$

where $\frac{F_{l+1} - F_l}{2b} \to \partial_s F (b \to 0)$ is used. Obviously, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}(u_{k+1} - u_k) = \frac{1}{2}\delta_k(u_{k+1} + u_k) - \frac{1}{2}(v_{k+1} + v_k)\left(z_{k+1} - z_k - y_{k+1} + y_k\right), \tag{3.17}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(v_{k+1} - v_k) = \frac{1}{2}\delta_k(v_{k+1} + v_k) - \frac{1}{2}(u_{k+1} + u_k)\left(z_{k+1} - z_k - y_{k+1} + y_k\right), \tag{3.18}$$

from (3.13) and (3.14), and

$$\frac{\mathrm{d}}{\mathrm{d}s}(x_{k+1} - x_k) = -\frac{1}{2}\left(u_{k+1}^2 - u_k^2 + v_{k+1}^2 - v_k^2\right), \tag{3.19}$$

by adding (3.15) and (3.16). Equation (3.19) coincides with equation (2.10).

Finally, in view of the relations (C.17)–(C.22), we have

$$\begin{aligned}
&\left(z_{k+1} - z_k - y_{k+1} + y_k\right)\delta_k \\
&= \left(z_{k+1} - z_k - y_{k+1} + y_k\right)\left(z_{k+1} - z_k + y_{k+1} - y_k\right) \\
&= -\frac{a^2}{4}\left[\left(\frac{\bar{f}_{k+1}\bar{f}_k}{f_{k+1}f_k} + \frac{f_{k+1}f_k}{\bar{f}_{k+1}\bar{f}_k}\right)^2 - \left(\frac{\bar{g}_{k+1}\bar{g}_k}{g_{k+1}g_k} + \frac{g_{k+1}g_k}{\bar{g}_{k+1}\bar{g}_k}\right)^2\right] \\
&= (u_{k+1} - u_k)(v_{k+1} - v_k).
\end{aligned} \tag{3.20}$$

A substitution of (3.20) into (3.17) and (3.18) yields (2.8) and (2.9).

From the construction of the fully discrete analogue of the CSP equation, the multi-soliton solution can be expressed in the following determinant form

$$u_{k,l} = \mathrm{i}\ln\left(\frac{\bar{f}_{k,l}\bar{g}_{k,l}}{f_{k,l}g_{k,l}}\right)_s = \frac{\mathrm{i}}{2}\left(\frac{\bar{f}'_{k,l}}{\bar{f}_{k,l}} + \frac{\bar{g}'_{k,l}}{\bar{g}_{k,l}} - \frac{f'_{k,l}}{f_{k,l}} - \frac{g'_{k,l}}{g_{k,l}}\right), \tag{3.21}$$

$$v_{k,l} = \mathrm{i}\ln\left(\frac{\bar{f}_{k,l}g_{k,l}}{f_{k,l}\bar{g}_{k,l}}\right)_s = \frac{\mathrm{i}}{2}\left(\frac{\bar{f}'_{k,l}}{\bar{f}_{k,l}} + \frac{g'_{k,l}}{g_{k,l}} - \frac{f'_{k,l}}{f_{k,l}} - \frac{\bar{g}'_{k,l}}{\bar{g}_{k,l}}\right), \tag{3.22}$$

$$y_{k,l} = ka - \left(\ln\left(f_{k,l}\bar{f}_{k,l}\right)\right)_s = ka - \frac{1}{2}\left(\frac{\bar{f}'_{k,l}}{\bar{f}_{k,l}} + \frac{f'_{k,l}}{f_{k,l}}\right), \tag{3.23}$$

$$z_{k,l} = ka - \left(\ln\left(g_{k,l}\bar{g}_{k,l}\right)\right)_s = ka - \frac{1}{2}\left(\frac{\bar{g}'_{k,l}}{\bar{g}_{k,l}} + \frac{g'_{k,l}}{g_{k,l}}\right), \tag{3.24}$$

thus

$$x_{k,l} = 2ka - \left(\ln\left(f_{k,l}\bar{f}_{k,l}g_{k,l}\bar{g}_{k,l}\right)\right)_s = 2ka - \frac{1}{2}\left(\frac{\bar{f}'_{k,l}}{\bar{f}_{k,l}} + \frac{\bar{g}'_{k,l}}{\bar{g}_{k,l}} + \frac{f'_{k,l}}{f_{k,l}} + \frac{g'_{k,l}}{g_{k,l}}\right). \tag{3.25}$$

Here

$$f_{k,l} = \tau_0(k, l),\ \bar{f}_{k,l} = \tau_1(k, l),\ g_{k,l} = \tau'_0(k, l),\ \bar{g}_{k,l} = \tau'_1(k, l), \tag{3.26}$$

$$f'_{k,l} = \rho_0(k, l),\ \bar{f}'_{k,l} = \rho_1(k, l),\ g'_{k,l} = \rho'_0(k, l),\ \bar{g}'_{k,l} = \rho'_1(k, l), \tag{3.27}$$

with $\tau_n(k, l)$ and $\tau'_n(k, l)$ defined the same as (3.1), and $\rho_n(k, l)$ and $\rho'_n(k, l)$ defined as

$$\rho_n(k, l) = \left|\phi^{(i)}_{(n+j-2)}(k, l)\right|_{1\leqslant i,j\leqslant N},\quad \rho'_n(k, l) = \left|\psi^{(i)}_{(n+j-3)}(k, l)\right|_{1\leqslant i,j\leqslant N}, \tag{3.28}$$

under the 2-reduction condition $q_i = -p_i$ ($i = 1, \cdots, N$ ).

**Remark 3.1.** Two intermediate variables $y_k$ and $z_k$ are introduced in constructing the fully discrete CSP equation. This often happens when we construct the full discretizations of a coupled system such as the coupled modified KdV equation [28].

## 4. Integrable self-adaptive moving mesh method

In this section, we propose a self-adaptive moving mesh method for the CSP equation (1.2)–(1.3) and demonstrate the advantage of this integrable scheme by performing several numerical experiments.

### *4.1. Numerical scheme*

One of the self-adaptive moving mesh methods for the CSP equation can be constructed by applying a semi-implicit Euler scheme to its integrable semi-discrete CSP equations (2.8)–(2.10). The resulting numerical scheme reads

$$p_k^{n+1} = p_k^n + \frac{\Delta t}{2}\delta_k^n\left(u_{k+1}^n + u_k^n\right) - \frac{\Delta t}{2\delta_k^n}p_k^n q_k^n\left(v_{k+1}^n + v_k^n\right), \tag{4.1}$$

$$q_k^{n+1} = q_k^n + \frac{\Delta t}{2}\delta_k^n\left(v_{k+1}^n + v_k^n\right) - \frac{\Delta t}{2\delta_k^n}p_k^n q_k^n\left(u_{k+1}^n + u_k^n\right), \tag{4.2}$$

$$\delta_k^{n+1} = -\frac{\Delta t}{2}\left(\left(u_{k+1}^{n+1}\right)^2 + \left(v_{k+1}^{n+1}\right)^2 - \left(u_k^{n+1}\right)^2 - \left(v_k^{n+1}\right)^2\right). \tag{4.3}$$

Here $p_k = u_{k+1} - u_k$, $q_k = v_{k+1} - v_k$, $\delta_k = x_{k+1} - x_k$. The superscript $n$ represents the numerical value at $t = n\Delta t$. Periodic boundary conditions are applied. For convenience, we reserve the time $t \to -t$ so that the left-moving wave becomes the right-moving one. In what follows, we report the numerical results for one- and two-soliton solutions.

### *4.2. Numerical experiments*

For the sake of numerical experiments, we list exact one- and two-soliton solutions for the continuous, semi- and fully-discrete CSP equation.

**(1). One-soliton solution**: the $\tau$-functions for the one-soliton solution of the CSP equation (1.2)–(1.3) are

$$f \propto 1 + \mathrm{i}\mathrm{e}^{\theta} \quad g \propto 1 + \mathrm{i}s_1\mathrm{e}^{\theta}, \tag{4.4}$$

where $s_1 = (1 - p_1)/(1 + p_1)$, $\theta = p_1 y + s/p_1 + y_0$. This leads to the one-soliton solution in parametric form

$$u(y, s) = \frac{1}{p_1}\left(\operatorname{sech}\theta + \operatorname{sgn}(s_1)\operatorname{sech}(\theta - \Delta)\right), \tag{4.5}$$

$$v(y, s) = \frac{1}{p_1}\left(\operatorname{sech}\theta - \operatorname{sgn}(s_1)\operatorname{sech}(\theta - \Delta)\right), \tag{4.6}$$

$$x = y - \frac{1}{p_1}(\tanh(\theta) + \tanh(\theta - \Delta)). \tag{4.7}$$

where $\exp(-\Delta) = |s_1|$.

For the semi-discrete CSP equation, the $\tau$-functions are

$$f_k \propto 1 + \mathrm{i}\left(\frac{1 + ap_1}{1 - ap_1}\right)^k \mathrm{e}^{s/p_1 + y_0}, \quad g_k \propto 1 + \mathrm{i}s_1\left(\frac{1 + ap_1}{1 - ap_1}\right)^k \mathrm{e}^{s/p_1 + y_0}. \tag{4.8}$$

Finally for the fully discrete CSP equation, the $\tau$-functions are

$$f_{k,l} \propto 1 + \mathrm{i}\left(\frac{1 + ap_1}{1 - ap_1}\right)^k \left(\frac{1 + bp_1^{-1}}{1 - bp_1^{-1}}\right)^l, \tag{4.9}$$

$$g_{k,l} \propto 1 + \mathrm{i}s_1\left(\frac{1 + ap_1}{1 - ap_1}\right)^k \left(\frac{1 + bp_1^{-1}}{1 - bp_1^{-1}}\right)^l. \tag{4.10}$$

The initial conditions for one-soliton propagation are taken from (4.4) with parameters $y_0 = 0$, and $p_1 = 0.9$, $p_1 = 2.0$. The initial profiles are shown in figures 1(a) and (b), respectively. The simulations are run on a domain $[-40, 40]$ with 800 grid points, thus the average mesh size is 0.1.

When $p_1 = 0.9$, $u$ is symmetric with two spikes, and $v$ is antisymmetric. The comparison between the numerical and analytical results is shown in figure 2, together with the non-uniform mesh. It can be seen that the non-uniform mesh is dense around the peak points of the solitons. Moreover, the most dense part of the non-uniform mesh moves along with the peak point. When $p_1 = 2.0$, $u$ is antisymmetric, and $v$ is symmetric with a

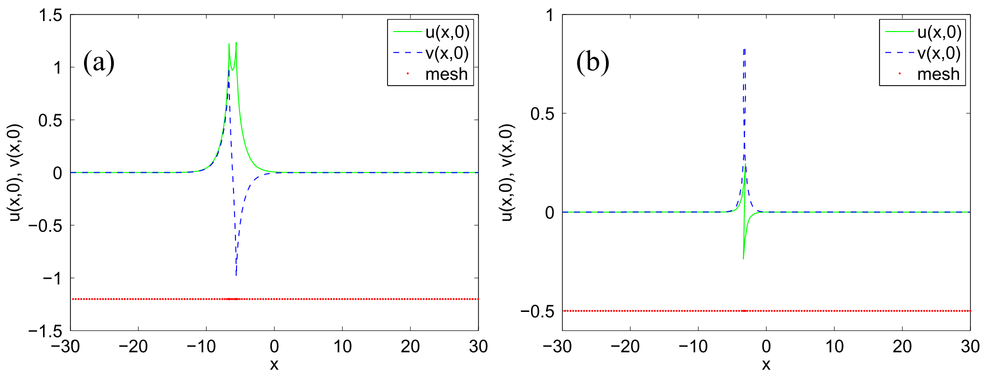


**Figure 1.** Initial conditions for CSP equation. (a) $p_1 = 0.9$; (b) $p_1 = 2.0$.

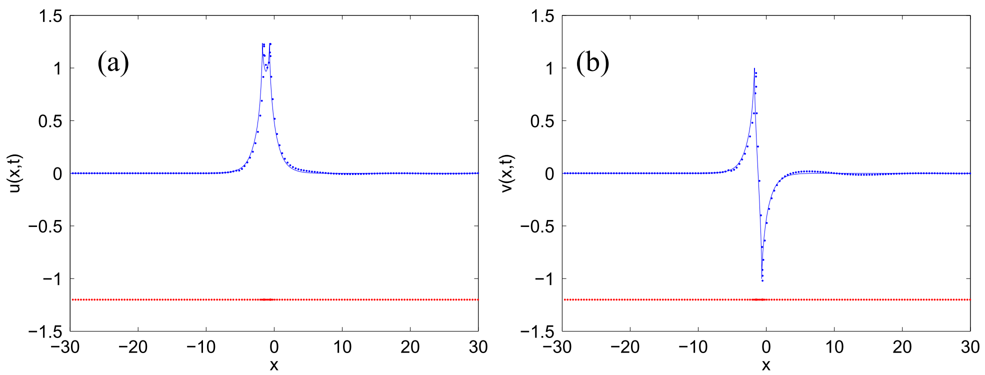


**Figure 2.** Comparison between numerical and analytical solutions for one-soliton solution to the CSP equation with $p_1 = 0.9$ at $t = 4.0$; solid line: analytical solution, blue dot: numerical solution, red dot: self-adaptive mesh; (a) profile of $u$, (b) profile of $v$.

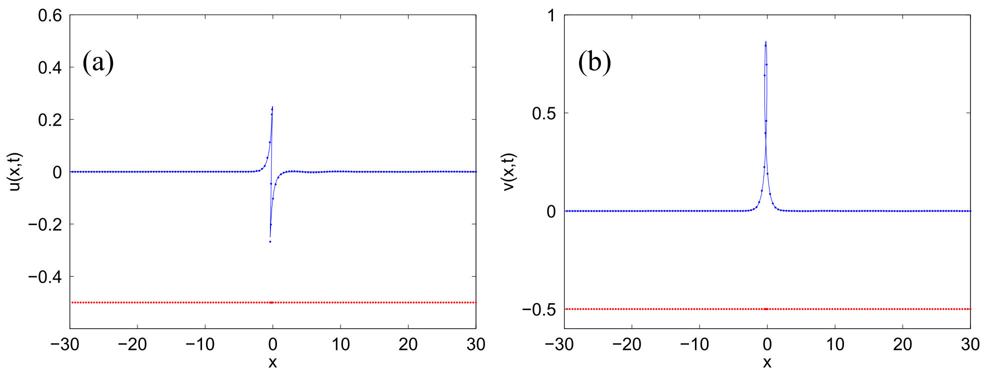


**Figure 3.** Comparison between numerical and analytical results for one-soliton solution to the CSP equation with $p_1 = 2.0$ at $t = 12.0$; solid line: analytical solution, blue dot: numerical solution, red dot: self-adaptive mesh; (a) profile of $u$, (b) profile of $v$.

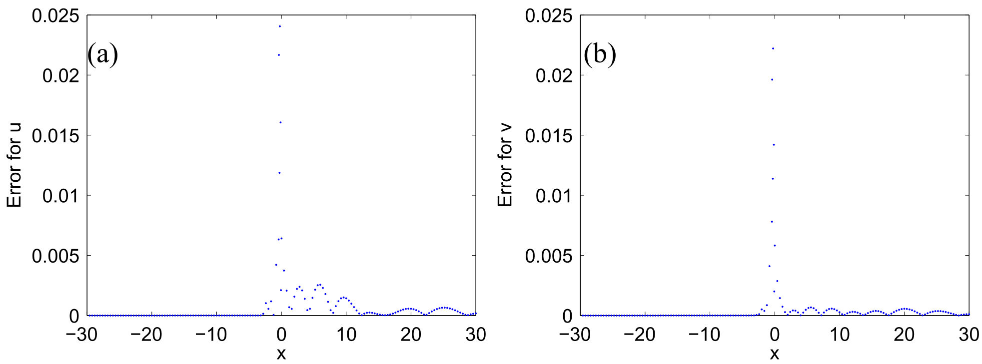


**Figure 4.** The error between numerical and analytical results for the one-soliton solution to the CSP equation with $p_1 = 2.0$ at $t = 12.0$; (a) error in $u$, (b) error in $v$.

loop structure. The comparison between the numerical and analytical results is shown in figure 3. The error between the numerical solution and the analytical one is displayed in figure 4.

(**2**). **Two-soliton solution**: the $\tau$-functions for two-soliton solution of the CSP equations (1.2) and (1.3) are

$$f \propto 1 + \mathrm{i}e^{\theta_1} + \mathrm{i}e^{\theta_2} - b_{12}e^{\theta_1+\theta_2}, \tag{4.11}$$

$$g \propto 1 + \mathrm{i}s_1e^{\theta_1} + \mathrm{i}s_2e^{\theta_2} - b'_{12}e^{\theta_1+\theta_2}, \tag{4.12}$$

with $s_i = (1 - p_i)/(1 + p_i)$, $\theta_i = p_i y + s/p_1 + y_{i0}$ $(i = 1, 2)$, and $b_{12} = (p_1 - p_2)^2/(p_1 + p_2)^2$, and $b'_{12} = b_{12}{}^*s_1{}^*s_2$.

For the semi-discrete CSP equation, the $\tau$-functions are

$$f_k \propto 1 + \mathrm{i}\left(\frac{1 + ap_1}{1 - ap_1}\right)^k e^{\xi_1} + \mathrm{i}\left(\frac{1 + ap_2}{1 - ap_2}\right)^k e^{\xi_2} - \left(\frac{(1 + ap_1)(1 + ap_2)}{(1 - ap_1)(1 - ap_2)}\right)^k b_{12}e^{\xi_1+\xi_2}, \tag{4.13}$$

$$g_k \propto 1 + \mathrm{i}s_1\left(\frac{1 + ap_1}{1 - ap_1}\right)^k e^{\xi_1} + \mathrm{i}s_2\left(\frac{1 + ap_2}{1 - ap_2}\right)^k e^{\xi_2} - b'_{12}\left(\frac{(1 + ap_1)(1 + ap_2)}{(1 - ap_1)(1 - ap_2)}\right)^k e^{\xi_1+\xi_2}, \tag{4.14}$$

with $p_i^{-1}s + y_{i0}$ $(i = 1, 2)$.

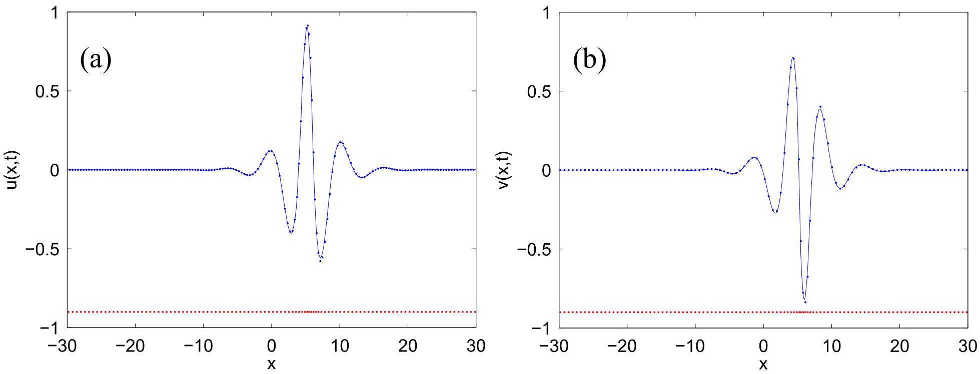


**Figure 5.** Comparison between numerical and analytical results for breather solution to the CSP equation for $p_1 = 0.4 + \mathrm{i}$, $p_2 = 0.4 - \mathrm{i}$ at $t = 10$; solid line: analytical solution, dashed line: numerical solution; (a) profile of $u$, (b) profile of $v$.

For the fully discrete CSP equation, the $\tau$-functions are

$$f_{k,l} \propto 1 + \mathrm{i}\left(\frac{1 + ap_1}{1 - ap_1}\right)^k\left(\frac{1 + bp_1^{-1}}{1 - bp_1^{-1}}\right)^l + \mathrm{i}\left(\frac{1 + ap_2}{1 - ap_2}\right)^k\left(\frac{1 + bp_2^{-1}}{1 - bp_2^{-1}}\right)^l - b_{12}\left(\frac{(1 + ap_1)(1 + ap_2)}{(1 - ap_1)(1 - ap_2)}\right)^k\left(\frac{(1 + bp_1^{-1})(1 + bp_2^{-1})}{(1 - bp_1^{-1})(1 - bp_2^{-1})}\right)^l, \tag{4.15}$$

$$g_{k,l} \propto 1 + \mathrm{i}s_1\left(\frac{1 + ap_1}{1 - ap_1}\right)^k\left(\frac{1 + bp_1^{-1}}{1 - bp_1^{-1}}\right)^l + \mathrm{i}s_2\left(\frac{1 + ap_2}{1 - ap_2}\right)^k\left(\frac{1 + bp_2^{-1}}{1 - bp_2^{-1}}\right)^l - b_{12}'\left(\frac{(1 + ap_1)(1 + ap_2)}{(1 - ap_1)(1 - ap_2)}\right)^k\left(\frac{(1 + bp_1^{-1})(1 + bp_2^{-1})}{(1 - bp_1^{-1})(1 - bp_2^{-1})}\right)^l. \tag{4.16}$$

As pointed out in [16], when $p_1$ and $p_2$ are complex conjugate to each other, the two-soliton solution becomes a breather solution. Equations (4.11) and (4.12) are used as initial conditions with parameters chosen as $p_1 = 0.4 + \mathrm{i}$, $p_2 = 0.4 - \mathrm{i}$, $y_{10} = y_{20} = 0$. The numerical results at $t = 10$ are displayed in figure 5 in comparison with the analytical solution. Here a grid point of 800 is used on the domain $[-40, 40]$, the time step size is taken as $\Delta t = 0.005$. The error between the numerical solution and the analytical one is displayed in figure 6. It can be seen that the numerical results are in good agreement with the analytical ones.

## 5. Conclusions

In this paper, we proposed integrable semi-discrete and fully discrete analogues of a coupled short pulse equation. The determinant formulae of $N$-soliton solutions for the semi-discrete and fully discrete analogues of the CSP equations are also presented. In the continuous limit, the fully discrete CSP equation converges to the semi-discrete CSP equation, then further converges to the continuous CSP equation.

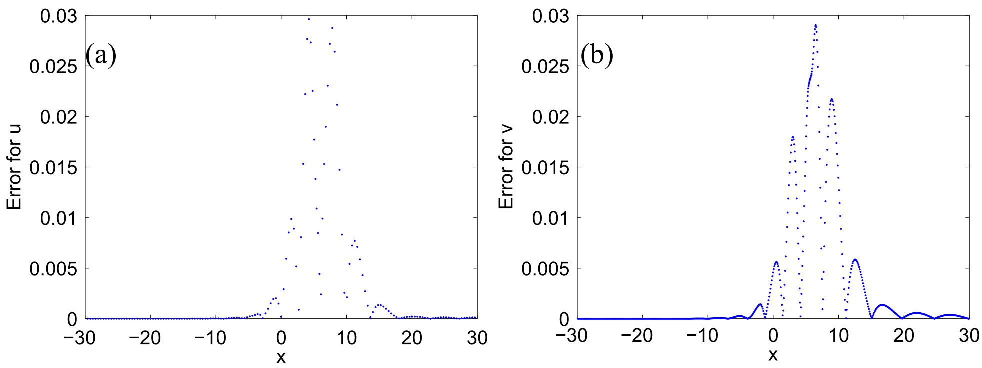


**Figure 6.** The error between numerical and analytical results for the one-breather solution to the CSP equation at $t = 10.0$; (a) error in $u$, (b) error in $v$.

In a series of papers by one of the authors, we have constructed integrable discretizations for a class of soliton equations with hodograph transformation, and successfully used them as self-adaptive moving mesh methods for the Camassa–Holm equation [30, 31] and the SP equation [16]. Based on the semi-discrete CSP equations (1.2) and (1.3), a self-adaptive moving mesh method is constructed and used for the numerical simulation of the CSP equation. It should be pointed out that the feature of self-adaptivity of the mesh is due to the hodograph transformation. In other words, the hodograph transformation converts the uniform and time-independent mesh into a non-uniform and time-dependent mesh. It is a further topic to seek this kind of self-adaptive moving method when the hodograph transformation is not present. The numerical results confirm that it is an excellent scheme due to the nature of integrability and self-adaptivity of the mesh. This is the first time this superior numerical method has been extended to a coupled system.

## Acknowledgments

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## Appendix A. Proof of proposition 1

**Proof.** For simplicity, we introduce a convenient notation

$$\left|0_k, 1_k, \cdots, N-1_k\right| = \begin{vmatrix} \phi_n^{(1)}(k) & \phi_{n+1}^{(1)}(k) & \cdots & \phi_{n+N-1}^{(1)}(k) \\ \phi_n^{(2)}(k) & \phi_{n+1}^{(2)}(k) & \cdots & \phi_{n+N-1}^{(2)}(k) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_n^{(N)}(k) & \phi_{n+1}^{(N)}(k) & \cdots & \phi_{n+N-1}^{(N)}(k) \end{vmatrix}, \tag{A.1}$$

$$\left|0'_k, 1'_k, \cdots, N-1'_k\right| = \begin{vmatrix} \psi_n^{(1)}(k) & \psi_{n+1}^{(1)}(k) & \cdots & \psi_{n+N-1}^{(1)}(k) \\ \psi_n^{(2)}(k) & \psi_{n+1}^{(2)}(k) & \cdots & \psi_{n+N-1}^{(2)}(k) \\ \cdots & \cdots & \cdots & \cdots \\ \psi_n^{(N)}(k) & \psi_{n+1}^{(N)}(k) & \cdots & \psi_{n+N-1}^{(N)}(k) \end{vmatrix}. \tag{A.2}$$

Since

$$\partial_{x_{-1}}\phi_n^{(i)}(k) = \phi_{n-1}^{(i)}(k), \;\; \phi_n^{(i)}(k+1) - \phi_n^{(i)}(k) = a\phi_{n+1}^{(i)}(k+1), \tag{A.3}$$

$$\partial_{x_{-1}}\psi_n^{(i)}(k) = \psi_{n-1}^{(i)}(k), \;\; \psi_n^{(i)}(k+1) - \psi_n^{(i)}(k) = a\psi_{n+1}^{(i)}(k+1), \tag{A.4}$$

$$\phi_n^{(i)}(k) - \psi_n^{(i)}(k) = \phi_{n+1}^{(i)}(k), \tag{A.5}$$

we can verify the following relations

$$\partial_{x_{-1}}\tau_n(k) = \left| -1_k,\, 1_k,\, \cdots,\, N-2_k,\, N-1_k \right|, \tag{A.6}$$

$$\begin{aligned}\tau_{n+1}(k+1) &= \left| 1_k,\, 2_k,\, \cdots,\, N-2_k,\, N-1_k,\, N_{k+1} \right| \\ &= \frac{1}{a}\left| 1_k,\, 2_k,\, \cdots,\, N-2_k,\, N-1_k,\, N-1_{k+1} \right|,\end{aligned} \tag{A.7}$$

$$\begin{aligned}\tau_n(k+1) &= \left| 0_{k+1},\, 1_{k+1},\, \cdots,\, N-2_{k+1},\, N-1_{k+1} \right| \\ &= \left| 0_k,\, 1_k,\, \cdots,\, N-2_k,\, N-1_{k+1} \right|,\end{aligned} \tag{A.8}$$

$$\begin{aligned}\partial_{x_{-1}}\tau_n(k+1) &= \left| -1_k,\, 1_k,\, \cdots,\, N-2_k,\, N-1_{k+1} \right| + \left| 0_k,\, 1_k,\, \cdots,\, N-2_k,\, N-2_{k+1} \right| \\ &= \left| -1_k,\, 1_k,\, \cdots,\, N-2_k,\, N-1_{k+1} \right| \\ &\quad + a\left| 0_k,\, 1_k,\, \cdots,\, N-2_k,\, N-1_{k+1} \right|.\end{aligned} \tag{A.9}$$

Combining (A.8) with (A.9), we have

$$\left(\frac{1}{a}\partial_{x_{-1}} - 1\right)\tau_n(k+1) = \frac{1}{a}\left| -1_k,\, 1_k,\, \cdots,\, N-2_k,\, N-1_{k+1} \right|. \tag{A.10}$$

Therefore, the Plücker relation for determinants

$$\begin{aligned}&\left| 0_k,\, 1_k,\, \cdots,\, N-2_k,\, N-1_k \right| \left| -1_k,\, 1_k,\, \cdots,\, N-2_k,\, N-1_{k+1} \right| \\ &\quad - \left| 0_k,\, 1_k,\, \cdots,\, N-2_k,\, N-1_{k+1} \right| \left| -1_k,\, 1_k,\, \cdots,\, N-2_k,\, N-1_k \right| \\ &\quad + \left| 1_k,\, \cdots,\, N-2_k,\, N-1_k,\, N-1_{k+1} \right| \left| -1_k,\, 0_k,\, 1_k,\, \cdots,\, N-2_k \right| = 0\end{aligned} \tag{A.11}$$

gives

$$\begin{aligned}&\left(\frac{1}{a}\partial_{x_{-1}} - 1\right)\tau_n(k+1) \times \tau_n(k) - \frac{1}{a}\tau_n(k+1) \\ &\quad \times \partial_{x_{-1}}\tau_n(k) + \tau_{n+1}(k+1)\tau_{n-1}(k) = 0,\end{aligned} \tag{A.12}$$

which is nothing but the bilinear equation (2.1). Equation (2.2) can be proved in the same way.

Now we proceed to the proof of equation (2.3). Similarly we can verify the following relations

$$\begin{aligned}\tau_n(k) &= \left| 0_k,\, 1_k,\, \cdots,\, N-2_k,\, N-1_k \right| \\ &= \left| 0'_k,\, 1'_k,\, \cdots,\, N-2'_k,\, N-1_k \right|,\end{aligned} \tag{A.13}$$

$$\begin{aligned}\tau_{n+1}(k) &= \left| 1'_k,\, 2'_k,\, \cdots,\, N-1'_k,\, N_k \right| \\ &= \left| 1'_k,\, 2'_k,\, \cdots,\, N-1'_k,\, N-1_k \right|,\end{aligned} \tag{A.14}$$

$$\left(\partial_{x_{-1}} - 1\right)\tau_n(k) = \left|-1'_k,\ 1'_k,\ \cdots,\ N-2'_k,\ N-1_k\right|. \tag{A.15}$$

Therefore the Plücker relation for determinants:

$$\begin{aligned}&\left|-1'_k,\ 1'_k,\ \cdots,\ N-2'_k,\ N-1_k\right|\left|0'_k,\ 1'_k,\ \cdots,\ N-2'_k,\ N-1'_k\right|\\ &\quad-\left|0'_k,\ 1'_k,\ \cdots,\ N-2'_k,\ N-1_k\right|\left|-1'_k,\ 1'_k,\ \cdots,\ N-2'_k,\ N-1'_k\right|\\ &\quad+\left|-1'_k,\ 0'_k,\ 1'_k,\ \cdots,\ N-2'_k\right|\left|1'_k,\ \cdots,\ N-1'_k,\ N-1_k\right| = 0,\end{aligned} \tag{A.16}$$

gives

$$\left(\partial_{x_{-1}} - 1\right)\tau_n(k) \times \tau'_n(k) - \tau_n(k) \times \partial_{x_{-1}}\tau'_n(k) + \tau_{n+1}(k)\tau'_{n-1}(k) = 0. \tag{A.17}$$

Therefore, equation (2.3) is proved. □

## Appendix B. Proof of theorem 1

**Proof.** By putting $s = 2x_{-1}$, $\tau_0(k) = f_k$, $\tau_1(k) = \bar{f}_k$, (2.1) can be converted into

$$\left(\frac{2}{a}D_s - 1\right)f_{k+1} \cdot f_k = -\bar{f}_{k+1}\bar{f}_k, \tag{B.1}$$

$$\left(\frac{2}{a}D_s - 1\right)\bar{f}_{k+1} \cdot \bar{f}_k = -f_{k+1}f_k, \tag{B.2}$$

while by putting $\tau'_0(k) = g_k$, $\tau'_1(k) = \bar{g}_k$, (2.2) can be converted into

$$\left(\frac{2}{a}D_s - 1\right)g_{k+1} \cdot g_k = -\bar{g}_{k+1}\bar{g}_k, \tag{B.3}$$

$$\left(\frac{2}{a}D_s - 1\right)\bar{g}_{k+1} \cdot \bar{g}_k = -g_{k+1}g_k. \tag{B.4}$$

Furthermore, the above bilinear equations can be rewritten as the following logarithmic derivatives

$$\frac{2}{a}\left(\ln\frac{f_{k+1}}{f_k}\right)_s - 1 = -\frac{\bar{f}_{k+1}\bar{f}_k}{f_{k+1}f_k}, \tag{B.5}$$

$$\frac{2}{a}\left(\ln\frac{\bar{f}_{k+1}}{\bar{f}_k}\right)_s - 1 = -\frac{f_{k+1}f_k}{\bar{f}_{k+1}\bar{f}_k}, \tag{B.6}$$

$$\frac{2}{a}\left(\ln\frac{g_{k+1}}{g_k}\right)_s - 1 = -\frac{\bar{g}_{k+1}\bar{g}_k}{g_{k+1}g_k}, \tag{B.7}$$

$$\frac{2}{a}\left(\ln\frac{\bar{g}_{k+1}}{\bar{g}_k}\right)_s - 1 = -\frac{g_{k+1}g_k}{\bar{g}_{k+1}\bar{g}_k}. \tag{B.8}$$

Introducing two intermediate variable transformations

$$\sigma_k(s) = 2\mathrm{i}\,\ln\left(\frac{\bar{f}_k(s)}{f_k(s)}\right), \quad \sigma'_k(s) = 2\mathrm{i}\,\ln\left(\frac{\bar{g}_k(s)}{g_k(s)}\right),$$

one arrives at a pair of semi-discrete sine-Gordon equations

$$\frac{1}{2a}(\sigma_{k+1} - \sigma_k)_s = \sin\left(\frac{\sigma_{k+1} + \sigma_k}{2}\right), \tag{B.9}$$

$$\frac{1}{2a}\left(\sigma'_{k+1} - \sigma'_k\right)_s = \sin\left(\frac{\sigma'_{k+1} + \sigma'_k}{2}\right). \tag{B.10}$$

It then follows that

$$\left(\cos\left(\frac{\sigma_{k+1} + \sigma_k}{2}\right)\right)_s = -\frac{1}{4a}\left((\sigma_{k+1,s})^2 - (\sigma_{k,s})^2\right), \tag{B.11}$$

$$\left(\cos\left(\frac{\sigma'_{k+1} + \sigma'_k}{2}\right)\right)_s = -\frac{1}{4a}\left(\left(\sigma'_{k+1,s}\right)^2 - \left(\sigma'_{k,s}\right)^2\right), \tag{B.12}$$

where $\sigma_{k,s}$ denotes the derivative of $\sigma_k$ with respective to $s$.

Next, we introduce dependent variable transformations

$$u_k = \frac{1}{2}\left(\sigma_{k,s} + \sigma'_{k,s}\right) = \mathrm{i}\,\ln\left(\frac{\bar{f}_k\,\bar{g}_k}{f_k\,g_k}\right)_s, \tag{B.13}$$

$$v_k = \frac{1}{2}\left(\sigma_{k,s} - \sigma'_{k,s}\right) = \mathrm{i}\,\ln\left(\frac{\bar{f}_k\,g_k}{f_k\,\bar{g}_k}\right)_s, \tag{B.14}$$

and discrete hodograph transformation

$$x_k = 2ka - \left(\ln\left(f_k\bar{f}_k\,g_k\bar{g}_k\right)\right)_s. \tag{B.15}$$

Then the non-uniform mesh can be derived as

$$\begin{aligned}
\delta_k &= x_{k+1} - x_k \\
&= 2a - \left(\ln\frac{f_{k+1}\bar{f}_{k+1}\,g_{k+1}\bar{g}_{k+1}}{f_k\bar{f}_k\,g_k\bar{g}_k}\right)_s \\
&= \frac{a}{2}\left(\frac{f_{k+1}\bar{f}_k}{\bar{f}_{k+1}f_k} + \frac{\bar{f}_{k+1}f_k}{f_{k+1}\bar{f}_k} + \frac{g_{k+1}\bar{g}_k}{\bar{g}_{k+1}g_k} + \frac{\bar{g}_{k+1}g_k}{g_{k+1}\bar{g}_k}\right) \\
&= a\left(\cos\left(\frac{\sigma_{k+1} + \sigma_k}{2}\right) + \cos\left(\frac{\sigma'_{k+1} + \sigma'_k}{2}\right)\right).
\end{aligned} \tag{B.16}$$

Taking the derivative with respect to $s$ results in

$$\begin{aligned}
\frac{\mathrm{d}\delta_k}{\mathrm{d}s} &= a\left(\cos\left(\frac{\sigma_{k+1} + \sigma_k}{2}\right)_s + \cos\left(\frac{\sigma'_{k+1} + \sigma'_k}{2}\right)_s\right) \\
&= -\frac{1}{2}\left(u_{k+1}^2 - u_k^2 + v_{k+1}^2 - v_k^2\right).
\end{aligned} \tag{B.17}$$

Furthermore, assuming

$$p_k = \sec\left(\frac{\sigma_k + \sigma_{k+1} + \sigma'_k + \sigma'_{k+1}}{4}\right), \quad q_k = \sec\left(\frac{\sigma_k + \sigma_{k+1} - \sigma'_k - \sigma'_{k+1}}{4}\right),$$

we have

$$\delta_k = \frac{2a}{p_k q_k}, \tag{B.18}$$

$$\begin{aligned}
\frac{dp_k^{-1}}{d\ s} &= \frac{\mathrm{d}}{\mathrm{d}s}\cos\left(\frac{\sigma_k + \sigma_{k+1} + \sigma'_k + \sigma'_{k+1}}{4}\right)\\
&= -\sin\left(\frac{\sigma_k + \sigma_{k+1} + \sigma'_k + \sigma'_{k+1}}{4}\right)\frac{u_k + u_{k+1}}{2}\\
&= -\frac{q_k}{2}\left(\sin\left(\frac{\sigma_k + \sigma_{k+1}}{2}\right) - \sin\left(\frac{\sigma'_k + \sigma'_{k+1}}{2}\right)\right)\frac{u_k + u_{k+1}}{2}\\
&= -\frac{q_k}{2}\frac{1}{2a}\left(\sigma_{k+1} - \sigma_k + \sigma'_{k+1} - \sigma'_k\right)_s\frac{u_k + u_{k+1}}{2}\\
&= -\frac{q_k}{4a}\left(u_{k+1}^2 - u_k^2\right),
\end{aligned} \tag{B.19}$$

and similarly

$$\frac{\mathrm{d}q_k^{-1}}{\mathrm{d}s} = -\frac{p_k}{4a}\left(v_{k+1}^2 - v_k^2\right). \tag{B.20}$$

Thus, in turn, equations (B.19) and (B.20) become

$$\frac{\mathrm{d}p_k^2}{\mathrm{d}s} = p_k^2\frac{u_{k+1}^2 - u_k^2}{\delta_k}, \tag{B.21}$$

and

$$\frac{\mathrm{d}q_k^2}{\mathrm{d}s} = q_k^2\frac{v_{k+1}^2 - v_k^2}{\delta_k}, \tag{B.22}$$

respectively.

On the other hand, with the help of trigonometric identity, $p_k^2$ can be expressed as

$$\begin{aligned}
p_k^2 &= 1 + \tan^2\left(\frac{\sigma_k + \sigma_{k+1} + \sigma'_k + \sigma'_{k+1}}{4}\right)\\
&= 1 + p_k^2\sin^2\left(\frac{\sigma_k + \sigma_{k+1} + \sigma'_k + \sigma'_{k+1}}{4}\right)\\
&= 1 + \frac{p_k^2 q_k^2}{4a^2}\left(u_{k+1}^2 - u_k^2\right)\\
&= 1 + \left(\frac{u_{k+1} - u_k}{\delta_k}\right)^2,
\end{aligned} \tag{B.23}$$

and similarly

$$q_k^2 = 1 + \left(\frac{v_{k+1} - v_k}{\delta_k}\right)^2. \tag{B.24}$$

Therefore, we finally have

$$\frac{d}{ds}\left(\frac{u_{k+1} - u_k}{\delta_k}\right) = \left(1 + \left(\frac{u_{k+1} - u_k}{\delta_k}\right)^2\right)\frac{u_{k+1} + u_k}{2}, \tag{B.25}$$

$$\frac{d}{ds}\left(\frac{v_{k+1} - v_k}{\delta_k}\right) = \left(1 + \left(\frac{v_{k+1} - v_k}{\delta_k}\right)^2\right)\frac{v_{k+1} + v_k}{2}. \tag{B.26}$$

Substituting (B.17) into (B.25) and (B.26), one arrives at (2.8) and (2.9), the first two equations of the semi-discrete CSP equation. □

## Appendix C. Proof of theorem 2

**Proof.** The bilinear equations (3.2)–(3.5) imply the following bilinear equations

$$\left(\frac{2}{a}D_s - 1\right)f_{k+1,l} \cdot f_{k,l} + \bar{f}_{k+1,l}\bar{f}_{k,l} = 0, \tag{C.1}$$

$$\left(\frac{2}{a}D_s - 1\right)\bar{f}_{k+1,l} \cdot \bar{f}_{k,l} + f_{k+1,l}f_{k,l} = 0, \tag{C.2}$$

$$\left(\frac{2}{a}D_s - 1\right)g_{k+1,l} \cdot g_{k,l} + \bar{g}_{k+1,l}\bar{g}_{k,l} = 0, \tag{C.3}$$

$$\left(\frac{2}{a}D_s - 1\right)\bar{g}_{k+1,l} \cdot \bar{g}_{k,l} + g_{k+1,l}g_{k,l} = 0, \tag{C.4}$$

$$(2bD_s - 1)f_{k,l+1} \cdot \bar{f}_{k,l} + f_{k,l}\bar{f}_{k,l+1} = 0, \tag{C.5}$$

$$(2bD_s - 1)\bar{f}_{k,l+1} \cdot f_{k,l} + \bar{f}_{k,l}f_{k,l+1} = 0, \tag{C.6}$$

$$(2bD_s - 1)g_{k,l+1} \cdot \bar{g}_{k,l} + g_{k,l}\bar{g}_{k,l+1} = 0, \tag{C.7}$$

$$(2bD_s - 1)\bar{g}_{k,l+1} \cdot g_{k,l} + \bar{g}_{k,l}g_{k,l+1} = 0, \tag{C.8}$$

which can be rewritten by logarithmic derivatives as

$$\frac{2}{a}\left(\ln\frac{f_{k+1,l}}{f_{k,l}}\right)_s = 1 - \frac{\bar{f}_{k+1,l}\bar{f}_{k,l}}{f_{k+1,l}f_{k,l}}, \tag{C.9}$$

$$\frac{2}{a}\left(\ln\frac{\bar{f}_{k+1,l}}{\bar{f}_{k,l}}\right)_s = 1 - \frac{f_{k+1,l}f_{k,l}}{\bar{f}_{k+1,l}\bar{f}_{k,l}}, \tag{C.10}$$

$$\frac{2}{a}\left(\ln\frac{g_{k+1,l}}{g_{k,l}}\right)_s = 1 - \frac{\bar{g}_{k+1,l}\bar{g}_{k,l}}{g_{k+1,l}g_{k,l}}, \tag{C.11}$$

$$\frac{2}{a}\left(\ln\frac{\bar{g}_{k+1,l}}{\bar{g}_{k,l}}\right)_s = 1 - \frac{g_{k+1,l}g_{k,l}}{\bar{g}_{k+1,l}\bar{g}_{k,l}}, \tag{C.12}$$

$$2b\left(\ln\frac{f_{k,l+1}}{\bar{f}_{k,l}}\right)_s = 1 - \frac{f_{k,l}\bar{f}_{k,l+1}}{f_{k,l+1}\bar{f}_{k,l}}, \tag{C.13}$$

$$2b\left(\ln\frac{\bar{f}_{k,l+1}}{f_{k,l}}\right)_s = 1 - \frac{\bar{f}_{k,l}f_{k,l+1}}{\bar{f}_{k,l+1}f_{k,l}} \tag{C.14}$$

$$2b\left(\ln\frac{g_{k,l+1}}{\bar{g}_{k,l}}\right)_s = 1 - \frac{g_{k,l}\bar{g}_{k,l+1}}{g_{k,l+1}\bar{g}_{k,l}}, \tag{C.15}$$

$$2b\left(\ln\frac{\bar{g}_{k,l+1}}{g_{k,l}}\right)_s = 1 - \frac{\bar{g}_{k,l}g_{k,l+1}}{\bar{g}_{k,l+1}g_{k,l}}. \tag{C.16}$$

Based on the dependent variable transformation (3.10) and discrete hodograph transformation (3.11), we can verify the following relations

$$u_{k+1,l} - u_{k,l} = \frac{\mathrm{i}a}{2}\left(\frac{\bar{f}_{k+1,l}\,\bar{f}_{k,l}}{f_{k+1,l}\,f_{k,l}} + \frac{\bar{g}_{k+1,l}\,\bar{g}_{k,l}}{g_{k+1,l}\,g_{k,l}} - \frac{f_{k+1,l}\,f_{k,l}}{\bar{f}_{k+1,l}\,\bar{f}_{k,l}} - \frac{g_{k+1,l}\,g_{k,l}}{\bar{g}_{k+1,l}\,\bar{g}_{k,l}}\right), \tag{C.17}$$

$$u_{k,l+1} + u_{k,l} = \frac{\mathrm{i}}{2b}\left(\frac{f_{k,l}\,\bar{f}_{k,l+1}}{f_{k,l+1}\,\bar{f}_{k,l}} + \frac{g_{k,l}\,\bar{g}_{k,l+1}}{g_{k,l+1}\,\bar{g}_{k,l}} - \frac{\bar{f}_{k,l}\,f_{k,l+1}}{\bar{f}_{k,l+1}\,f_{k,l}} - \frac{\bar{g}_{k,l}\,g_{k,l+1}}{\bar{g}_{k,l+1}\,g_{k,l}}\right), \tag{C.18}$$

$$v_{k+1,l} - v_{k,l} = \frac{\mathrm{i}a}{2}\left(\frac{\bar{f}_{k+1,l}\,\bar{f}_{k,l}}{f_{k+1,l}\,f_{k,l}} - \frac{\bar{g}_{k+1,l}\,\bar{g}_{k,l}}{g_{k+1,l}\,g_{k,l}} - \frac{f_{k+1,l}\,f_{k,l}}{\bar{f}_{k+1,l}\,\bar{f}_{k,l}} + \frac{g_{k+1,l}\,g_{k,l}}{\bar{g}_{k+1,l}\,\bar{g}_{k,l}}\right), \tag{C.19}$$

$$v_{k,l+1} + v_{k,l} = \frac{\mathrm{i}}{2b}\left(\frac{f_{k,l}\,\bar{f}_{k,l+1}}{f_{k,l+1}\,\bar{f}_{k,l}} - \frac{g_{k,l}\,\bar{g}_{k,l+1}}{g_{k,l+1}\,\bar{g}_{k,l}} - \frac{\bar{f}_{k,l}\,f_{k,l+1}}{\bar{f}_{k,l+1}\,f_{k,l}} + \frac{\bar{g}_{k,l}\,g_{k,l+1}}{\bar{g}_{k,l+1}\,g_{k,l}}\right), \tag{C.20}$$

$$y_{k+1,l} - y_{k,l} = \frac{a}{2}\left(\frac{\bar{f}_{k+1,l}\,\bar{f}_{k,l}}{f_{k+1,l}\,f_{k,l}} + \frac{f_{k+1,l}\,f_{k,l}}{\bar{f}_{k+1,l}\,\bar{f}_{k,l}}\right), \tag{C.21}$$

$$z_{k+1,l} - z_{k,l} = \frac{a}{2}\left(\frac{\bar{g}_{k+1,l}\,\bar{g}_{k,l}}{g_{k+1,l}\,g_{k,l}} + \frac{g_{k+1,l}\,g_{k,l}}{\bar{g}_{k+1,l}\,\bar{g}_{k,l}}\right), \tag{C.22}$$

$$y_{k,l+1} - y_{k,l} = -\frac{1}{b} + \frac{1}{2b}\left(\frac{f_{k,l}\,\bar{f}_{k,l+1}}{f_{k,l+1}\,\bar{f}_{k,l}} + \frac{\bar{f}_{k,l}\,f_{k,l+1}}{\bar{f}_{k,l+1}\,f_{k,l}}\right), \tag{C.23}$$

$$z_{k,l+1} - z_{k,l} = -\frac{1}{b} + \frac{1}{2b}\left(\frac{g_{k,l}\,\bar{g}_{k,l+1}}{g_{k,l+1}\,\bar{g}_{k,l}} + \frac{\bar{g}_{k,l}\,g_{k,l+1}}{\bar{g}_{k,l+1}\,g_{k,l}}\right). \tag{C.24}$$

Then, the ratios on the rhs of equations (C.17)–(C.24) can be solved as

$$\frac{\bar{f}_{k+1,l}\,\bar{f}_{k,l}}{f_{k+1,l}\,f_{k,l}} = \frac{1}{a}\left[y_{k+1,l} - y_{k,l} - \frac{\mathrm{i}}{2}(u_{k+1,l} - u_{k,l} + v_{k+1,l} - v_{k,l})\right], \tag{C.25}$$

$$\frac{f_{k+1,l}\,f_{k,l}}{\bar{f}_{k+1,l}\,\bar{f}_{k,l}} = \frac{1}{a}\left[y_{k+1,l} - y_{k,l} + \frac{\mathrm{i}}{2}(u_{k+1,l} - u_{k,l} + v_{k+1,l} - v_{k,l})\right], \tag{C.26}$$

$$\frac{f_{k,l}\,\bar{f}_{k,l+1}}{f_{k,l+1}\,\bar{f}_{k,l}} = 1 + b\left[y_{k,l+1} - y_{k,l} - \frac{\mathrm{i}}{2}(u_{k,l+1} + u_{k,l} + v_{k,l+1} + v_{k,l})\right], \tag{C.27}$$

$$\frac{\bar{f}_{k,l}\,f_{k,l+1}}{\bar{f}_{k,l+1}\,f_{k,l}} = 1 + b\left[y_{k,l+1} - y_{k,l} + \frac{\mathrm{i}}{2}(u_{k,l+1} + u_{k,l} + v_{k,l+1} + v_{k,l})\right], \tag{C.28}$$

$$\frac{\bar{g}_{k+1,l}\,\bar{g}_{k,l}}{g_{k+1,l}\,g_{k,l}} = \frac{1}{a}\left[z_{k+1,l} - z_{k,l} - \frac{\mathrm{i}}{2}(u_{k+1,l} - u_{k,l} - v_{k+1,l} + v_{k,l})\right], \tag{C.29}$$

$$\frac{g_{k+1,l}\,g_{k,l}}{\bar{g}_{k+1,l}\,\bar{g}_{k,l}} = \frac{1}{a}\left[z_{k+1,l} - z_{k,l} + \frac{\mathrm{i}}{2}(u_{k+1,l} - u_{k,l} - v_{k+1,l} + v_{k,l})\right], \tag{C.30}$$

$$\frac{g_{k,l}\,\bar{g}_{k,l+1}}{g_{k,l+1}\,\bar{g}_{k,l}} = 1 + b\left[z_{k,l+1} - z_{k,l} - \frac{\mathrm{i}}{2}(u_{k,l+1} + u_{k,l} - v_{k,l+1} - v_{k,l})\right], \tag{C.31}$$

$$\frac{\bar{g}_{k,l}\,g_{k,l+1}}{\bar{g}_{k,l+1}\,g_{k,l}} = 1 + b\left[z_{k,l+1} - z_{k,l} + \frac{\mathrm{i}}{2}(u_{k,l+1} + u_{k,l} - v_{k,l+1} - v_{k,l})\right]. \tag{C.32}$$

By making a shift of $l \to l{+}1$ in (C.25), then dividing it by (C.26), meanwhile, dividing (C.27) by (C.28) after a shift of $k \to k{+}1$, one obtains

$$\begin{aligned}&\frac{y_{k+1,l+1} - y_{k,l+1} - \frac{\mathrm{i}}{2}(u_{k+1,l+1} - u_{k,l+1} + v_{k+1,l+1} - v_{k,l+1})}{y_{k+1,l} - y_{k,l} + \frac{\mathrm{i}}{2}(u_{k+1,l} - u_{k,l} + v_{k+1,l} - v_{k,l})}\\ &\quad= \frac{1 + b\left[y_{k+1,l+1} - y_{k+1,l} - \frac{\mathrm{i}}{2}(u_{k+1,l+1} + u_{k+1,l} + v_{k+1,l+1} + v_{k+1,l})\right]}{1 + b\left[y_{k,l+1} - y_{k,l} + \frac{\mathrm{i}}{2}(u_{k,l+1} + u_{k,l} + v_{k,l+1} + v_{k,l})\right]}.\end{aligned} \tag{C.33}$$

Similarly, one can obtain

$$\begin{aligned}&\frac{z_{k+1,l+1} - z_{k,l+1} - \frac{\mathrm{i}}{2}(u_{k+1,l+1} - u_{k,l} - v_{k+1,l+1} + v_{k,l+1})}{z_{k+1,l} - z_{k,l} + \frac{\mathrm{i}}{2}(u_{k+1,l} - u_{k,l} - v_{k+1,l} + v_{k,l})}\\ &\quad= \frac{1 + b\left[z_{k+1,l+1} - z_{k+1,l} - \frac{\mathrm{i}}{2}(u_{k+1,l+1} + u_{k+1,l} - v_{k+1,l+1} - v_{k+1,l})\right]}{1 + b\left[z_{k,l+1} - z_{k,l} + \frac{\mathrm{i}}{2}(u_{k,l+1} + u_{k,l} - v_{k,l+1} - v_{k,l})\right]},\end{aligned} \tag{C.34}$$

from relations (C.29)–(C.31). Equating the real parts and imaginary parts of (C.33) and (C.34), we have the fully discrete CSP equations (3.6)–(3.9). □

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