APPLICATIONS OF FRACTIONAL EXTERIOR DIFFERENTIAL IN THREE-DIMENSIONAL SPACE[*]

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Abstract : A brief survey of fractional calculus and fractional differential forms was firstly given. The fractional exterior transition to curvilinear coordinate at the origin were discussed and the two coordinate transformations for the fractional differentials for three-dimensional Cartesian coordinates to spherical and cylindrical coordinates are obtained , respectively. In particular , for $v = m = 1$, the usual exterior transformations , between the spherical coordinate and Cartesian coordinate , as well as the cylindrical coordinate and Cartesian coordinate , are found respectively , from fractional exterior transformation.

Key words : fractional differential form; Cartesian coordinate ; spherical coordinate ; cylindrical coordinate

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Introduction

In generalized integration and differentiation the question of extension of meaning is : can the meaning of derivatives of integral order $\mathrm{d}^n y / \mathrm{d} x^n$ be extended to have meaning where n is any number (e. g. , irrational , fraction or complex) ? In 1695 Leibniz invented above notation. Eular and Fourier mentioned derivatives of arbitrary order but they gave no applications or examples. So the honor of making the first application belongs to Abel in 1823. Abel applied the fractional calculus in the solution of an integral equation which arises in the formulation of the Tautochrone problem. Abel's solution was so elegant that it attempted the attention of Liouville who made the first major attempt to give a logical definition of a fractional derivative in 1832. Riemann in 1847 while a student wrote a paper published posthumously in which he gave a definition of a fractional operation. A definition named in honor of Riemann and Liouville is

$$\mathrm{d} x_i = \sum_{i=1}^{n} \mathrm{d} x_i \frac{\partial^v}{(\partial (x_i - a_i))^v}.$$

By now scientists and applied mathematicians have found the fractional calculus useful in various fields : rheology , quantitative biology , electrochemistry , scattering theory , diffusion , transport theory , probability potential theory and elasticity[1]. In recent years exterior calculus has been generalized by basing it on various graded algebras[2,3]. Other attempts at generalization are based on nonassociative geometries[4,5]. Recently , Cottrill-Shepherd and Naber gave the definition of a fractional exterior derivative[6] and found that fractional differential formal space generates new vector spaces of finite and infinite dimension , the definition of closed and exact forms are extended to the new fractional form spaces with closure and integrability condition worked out for a special case. Coordinate transformation rules are also computed.

1 Transition to Curvilinear Coordinates and Two Important Examples

In the paper the Rimmann-Liouville definition of fractional integration and differentiation will be used. $\Gamma(q)$ is the gamma function (generalized factorial) of the parameter " q " [i. e. , $\Gamma(n+1) = n\,!$ for all whole number , " n "] ,

$$\frac{\partial^q f(x)}{(\partial(x-a))^q} = \frac{1}{\Gamma(-q)} \int_a^x \frac{f(\xi)\,\mathrm{d}\xi}{(x-\xi)^{q+1}} \qquad (\mathrm{Re}(q) < 0) , \tag{1}$$

$$\frac{\partial^q f(x)}{(\partial(x-a))^q} = \frac{\partial^n}{\partial x^n}\left[\frac{1}{\Gamma(n-q)} \int_a^x \frac{f(\xi)\,\mathrm{d}\xi}{(x-\xi)^{q-n+1}}\right] \qquad (\frac{\mathrm{Re}(q) \geq 0}{n > q\,(n \text{ is whole})}). \tag{2}$$

The parameter q is the order of the integral or derivative and allowed to be complex. Positive real values of q represent derivatives and negative real values represent integrals. Eq. (1) is a fractional integral and Eq. (2) is a fractional derivative. In this paper , only real and positive values of q will be considered. If the partial derivative are allowed to assume fractional orders , a fractional exterior derivative can be defined

$$\mathrm{d}^v = \sum_{i=1}^{n} \mathrm{d}x_i \frac{\partial^v}{(\partial(x_i-a_i))^v}. \tag{3}$$

Note that the subscript i denotes the coordinate number , the superscript v denotes the order of the fractional coordinate differential , and a_i is the initial point of the derivative. For convenience , the initial point a_i for the fractional derivative is taken to the origin.

Let $\{x_i\}$ and $\{y_i\}$ be two coordinate systems with a one to one mapping between them in some neighborhood of $p \in E^n$. Take $\{x_i\}$ again to be Cartesian coordinates and $\{y_i\}$ to be curvilinear coordinates. Assume the $\{x_i\}$ can be written smoothly in terms of the $\{y_i\}$,

$$x_i = x_i(y). \tag{4}$$

The exterior derivative is then applied to Eq. (4) giving the following :

$$\mathrm{d}x_i = \sum_{i=1}^{n} \mathrm{d}y_l \frac{\partial x_i}{\partial y_l}. \tag{5}$$

In the two coordinate systems , the fractional exterior derivative d^v takes the following forms :

$$\mathrm{d}^v = \sum_{i=1}^{n} \mathrm{d}x_i \frac{\partial^v}{\partial x_i^v} \tag{6}$$

and

$$\mathrm{d}^v = \sum_{i=1}^{n} \mathrm{d}y_i \frac{\partial^v}{\partial y_i^v} \tag{7}$$

which gives rise to

$$\sum_{i=1}^{n} d x_i \frac{\partial^v}{\partial x_i^v} = \sum_{i=1}^{n} d y_i \frac{\partial^v}{\partial y_i^v}. \tag{8}$$

Consider a function f_k that maps points in E^n into the complex numbers

$$f_k = \frac{(1)}{(v+1)} (\prod_{i=1/i \ne k}^{n} x_i)^{v-m} x_k^v. \tag{9}$$

It is easily seen that

$$\frac{\partial^v f_k}{(\partial x_i)^v} = 0 \qquad (i \ne k, \text{ i.e. }, f_k \in \text{Ker}\left(\frac{\partial^v}{(\partial x_i)^v}\right)) \tag{10}$$

and

$$\frac{\partial^v f_k}{(\partial x_i)^v} = 1 \qquad (i = k). \tag{11}$$

Applying the fractional exterior derivative (8) to both sides of (9) in two different coordinate systems the following coordinate transformation rule can be obtained

$$d x_k^v = \sum_{i=1}^{n} \frac{d y_i^v}{(v+1)} \frac{\partial^v}{(\partial y_i)^v} ((\prod_{j=1/j \ne k}^{n} x_j(y))^{v-m} x_k(y)^v). \tag{12}$$

In what follows we consider the coordinate transformation for three-dimensional Cartesian to cylindrical coordinates and spherical coordinates.

Example **1**   Spherical coordinate

Consider the coordinate transformation of spherical coordinate

$$\begin{cases} x_1 = r \sin(\ ) \cos(\phi) \\ x_2 = r \sin(\ ) \sin(\phi) \qquad (r \ge 0, 0 \le \le, 0 \le \phi \le). \\ x_3 = r \cos(\ ) \end{cases} \tag{13}$$

The coordinate transformations for the fractional differentials are then

$$d x_1^v = \frac{(3v-2m+1)}{(v+1)(2v-2m+1)} \frac{\cos^{v-m}(\ )\cos^v(\phi)}{\sin^{m-2v}(\ )\sin^{m-v}(\phi)} r^{2v-2m} d r^v +$$
$$\frac{r^{3v-2m}}{(v+1)} \frac{\partial^v}{(\partial\ )^v}\left[\frac{\cos^{v-m}(\ )\cos^v(\phi)}{\sin^{m-2v}(\ )\sin^{m-v}(\phi)}\right] d\ ^v +$$
$$\frac{r^{3v-2m}}{(v+1)} \frac{\partial^v}{(\partial\phi)^v}\left[\frac{\cos^{v-m}(\ )\cos^v(\phi)}{\sin^{m-2v}(\ )\sin^{m-v}(\phi)}\right] d\phi^v, \tag{14}$$

$$d x_2^v = \frac{(3v-2m+1)}{(v+1)(2v-2m+1)} \frac{\cos^{v-m}(\ )\cos^{v-m}(\phi)\sin^v(\phi)}{\sin^{m-2v}(\ )} r^{2v-2m} d r^v +$$
$$\frac{r^{3v-2m}}{(v+1)} \frac{\partial^v}{(\partial\ )^v}\left[\frac{\cos^{v-m}(\ )\cos^{v-m}(\phi)\sin^v(\phi)}{\sin^{m-2v}(\ )}\right] d\ ^v +$$
$$\frac{r^{3v-2m}}{(v+1)} \frac{\partial^v}{(\partial\phi)^v}\left[\frac{\cos^{v-m}(\ )\cos^{v-m}(\phi)\sin^v(\phi)}{\sin^{m-2v}(\ )}\right] d\phi^v, \tag{15}$$

$$d x_3^v = \frac{(3v-2m+1)}{(v+1)(2v-2m+1)} \frac{\cos^{v-m}(\phi)\cos^v(\ )}{\sin^{m-v}(\phi)\sin^{2m-2v}(\ )} r^{2v-2m} d r^v +$$
$$\frac{r^{3v-2m}}{(v+1)} \frac{\partial^v}{(\partial\ )^v}\left[\frac{\cos^{v-m}(\phi)\cos^v(\ )}{\sin^{m-v}(\phi)\sin^{2m-2v}(\ )}\right] d\ ^v +$$
$$\frac{r^{3v-2m}}{(v+1)} \frac{\partial^v}{(\partial\phi)^v}\left[\frac{\cos^{v-m}(\phi)\cos^v(\ )}{\sin^{m-v}(\phi)\sin^{2m-2v}(\ )}\right] d\phi^v. \tag{16}$$

For $v = m = 1$, we from (14) - (16) have

$$\begin{cases} dx_1 = \sin(\ )\cos(\phi)\,dr + r\cos(\ )\cos(\phi)\,d\ - r\sin(\ )\sin(\phi)\,d\phi, \\ dx_2 = \sin(\ )\sin(\phi)\,dr + r\cos(\ )\sin(\phi)\,d\ + r\sin(\ )\cos(\phi)\,d\phi, \\ dx_3 = \cos(\ )\,dr - r\sin(\ )\,d\ , \end{cases} \quad (17)$$

that is,

$$\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} \sin(\ )\cos(\phi) & r\cos(\ )\cos(\phi) & -r\sin(\ )\sin(\phi) \\ \sin(\ )\sin(\phi) & r\cos(\ )\sin(\phi) & r\sin(\ )\cos(\phi) \\ \cos(\ ) & -r\sin(\phi) & 0 \end{bmatrix} \begin{bmatrix} dr \\ d\ \\ d\phi \end{bmatrix} \quad (18)$$

which is the same as the usuall exterior transformation between the spherical coordinate and Cartesian coordinate in three-dimensional space.

Example **2** Cylindrical coordinate

Consider the coordinate transformation of cylindrinal coordinate

$$\begin{cases} x_1 = r\cos(\phi), \\ x_2 = r\sin(\phi), \\ x_3 = z. \end{cases} \quad (19)$$

From (12), it is easy to see that the coordinate transformations for the fractional differentials are

$$\begin{aligned} dx_1^v = &\frac{(2v-m+1)}{(v+1)(v-2m+1)}\frac{z^{v-m}\cos^v(\phi)}{\sin^{m-v}(\phi)}r^{v-m}\,dr^v + \\ &\frac{r^{2v-m}z^{v-m}}{(v+1)}\frac{\partial^v}{(\partial\phi)^v}\left(\frac{\cos^v(\phi)}{\sin^{m-v}(\phi)}\right)d\phi^v + \\ &\frac{r^{2v-m}}{z^m}\frac{(v-m+1)}{(1-m)(v+1)}\frac{\cos^v(\phi)}{\sin^{m-v}(\phi)}dz^v, \end{aligned} \quad (20)$$

$$\begin{aligned} dx_2^v = &\frac{(2v-m+1)}{(v+1)(v-2m+1)}\frac{z^{v-m}\sin^v(\phi)}{\cos^{m-v}(\phi)}r^{v-m}\,dr^v + \\ &\frac{r^{2v-m}z^{v-m}}{(v+1)}\frac{\partial^v}{(\partial\phi)^v}\left(\frac{\sin^v(\phi)}{\cos^{m-v}(\phi)}\right)d\ ^v + \\ &\frac{r^{2v-m}}{z^m}\frac{(v-m+1)}{(1-m)(v+1)}\left(\frac{\sin^v(\phi)}{\cos^{m-v}(\phi)}\right)dz^v, \end{aligned} \quad (21)$$

$$\begin{aligned} dx_3^v = &\frac{(2v-m+1)}{(v+1)(v-2m+1)}\frac{z^{v-m}\cos^{v-m}(\phi)}{\sin^{m-v}(\phi)}r^{v-2m}\,dr^v + \\ &\frac{r^{2v-2m}z^v}{(v+1)}\frac{\partial^v}{(\partial\phi)^v}\left(\frac{\cos^{v-m}(\phi)}{\sin^{m-v}(\phi)}\right)d\phi^v + \\ &\frac{r^{2v-2m}\cos^{v-m}(\phi)}{\sin^{m-v}(\phi)}dz^v. \end{aligned} \quad (22)$$

For $v = m = 1$, we from (20) - (22) have

$$\begin{cases} dx_1 = \cos(\phi)\,dr - r\sin(\phi)\,d\phi, \\ dx_2 = \sin(\phi)\,dr + r\cos(\phi)\,d\phi, \\ dx_3 = dz, \end{cases} \quad (23)$$

that is,

$$
\begin{pmatrix} \mathrm{d}\,x_1 \\ \mathrm{d}\,x_2 \\ \mathrm{d}\,x_3 \end{pmatrix} = \begin{pmatrix} \cos(\phi) & -r\sin(\phi) & 0 \\ \sin(\phi) & r\cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}\,r \\ \mathrm{d}\,\phi \\ \mathrm{d}\,z \end{pmatrix}
\tag{24}
$$

which is the same as the usual exterior transformation between the spherical coordinate and Cartesian coordinate in three-dimensional space.

　　In summary , we have found the two coordinate transformations for the fractional differentials for three-dimensional Cartesian coordinates to spherical and cylindrical coordinates. In particular , for $m = v = 1$, the two above-mentioned coordinate transformations are the same as the standard results obtained from the exterior calculus.

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