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Differential Invariants of the (2+1)-Dimensional Breaking Soliton Equation

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Abstract: We construct the differential invariants of Lie symmetry pseudogroups of the (2+1)-dimensional breaking soliton equation and analyze the structure of the induced differential invariant algebra. Their syzygies and recurrence relations are classified. In addition, a moving frame and the invariantization of the breaking soliton equation are also presented. The algorithms are based on the method of equivariant moving frames.

Keywords: (2+1)-Dimensional Breaking Soliton Equation; Differential Invariants; Lie Pseudogroup; Moving Frames; Symmetry Group.

1 Introduction

The applications of differential invariants can be found in a broad range of problems arising in geometry, differential equations, mathematical physics, and engineering [1–9]. The determination of the structure of the algebra of differential invariants for a given Lie group or pseudogroup action is an essential first step in performing these applications. Fels and Olver [10, 11] introduced a new equivariant formulation of Cartan's method of moving frames, which then developed through a series of papers [12–15]. For a system of differential equations, the moving frame techniques have been used to obtain the structure equations and differential invariants for its symmetry groups directly from the infinitesimal determining equations [12, 16]. These algorithms are powerful and efficient for classifying the differential invariants as well as analyzing the induced algebra structure. In addition, they require only linear algebra and differentiation and do not require any explicit formulas for the moving frame, the differential invariants and invariant differential operators, or even the Maurer-Cartan forms.

The (2+1)-dimensional breaking soliton equation [17–19],

$$u_{tx} - 4u_x u_{xy} - 2u_{xx}u_y - u_{xxxy} = 0, \tag{1}$$

describes the (2+1)-dimensional interaction of a Riemann wave propagating along the y axis with a long wave along the x axis. This equation admits breaking solitons [20], and it becomes to the KdV equation when $y=x$. In recent years, a large number of papers have been focusing on Painlevé property, dromion-like structures, Lax pairs, and various exact solutions of this equation [21–30]. Yet, to the best of our knowledge, the differential invariants of (1) have not been studied so far. The goal of this paper is to investigate the algebra of differential invariants for (1) through the constructive computational algorithms [11, 14, 16]. We analyze the structure of the induced differential invariants algebra in detail and classified the syzygies and recurrence relations among the differential invariants.

The outline of this paper is as follows. In Section 2, the preliminaries about the algorithms used are presented. The detailed constructions of the algebra of differential invariants for the breaking soliton equation are given in Section 3, and the recurrence formulas and syzygies among them are also established. In Section 4, a moving frame for the breaking soliton equation is presented. Lastly, Section 5 presents a short summary and discussion.

2 Preliminaries

In this section, the theoretical preliminaries about the algorithms are introduced briefly [2, 3, 10, 11, 14, 16]. Consider a system of differential equations with p independent variables $x=(x^1, \cdots, x^p)$ and q dependent variables $u=(u^1, \cdots, u^q)$, and the derivatives u_J^α up to some finite order n, which reads

$$\Delta_\nu(x, u^{(n)})=0, \nu=1, 2, \cdots, k. \tag{2}$$

Here, $z=(x, u)$ is regarded as local coordinates on the total space M, a manifold of dimension $m=p+q$.

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Consider a smooth vector field on M,

$$\boldsymbol{v}=\sum_{i=1}^{p}\xi^i(x,u)\frac{\partial}{\partial x^i}+\sum_{\alpha=1}^{q}\varphi^\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{3}$$

And let

$$\boldsymbol{v}^{(n)}=\sum_{i=1}^{p}\xi^i\frac{\partial}{\partial x^i}+\sum_{\alpha=1}^{q}\sum_{\#J=0}^{n}\hat{\varphi}_J^\alpha\frac{\partial}{\partial u_J^\alpha}, \tag{4}$$

which denotes its nth-order prolongation to $J^n(M, p)$, whose coefficients are determined by the well-known prolongation formula

$$\hat{\varphi}_J^\alpha=D_J\left(\varphi^\alpha-\sum_{i=1}^{p}u_i^\alpha\xi^i\right)+\sum_{i=1}^{p}u_{J,i}^\alpha\xi^i. \tag{5}$$

A vector field $\mathbf{v}$ is an infinitesimal symmetry of the system of the differential equation (2) if and only if it satisfies the infinitesimal invariance condition. When expanded, this forms the following system of infinitesimal determining equation,

$$\mathcal{L}(\cdots,x^i,\cdots,u^\alpha,\cdots,\xi_A^i,\cdots,\varphi_A^\alpha,\cdots)=0, \tag{6}$$

which includes the original determining equations along with all equations obtained by repeated differentiation.

Let $\mathcal{H}^{(n)}\to J^n(M, p)$ be the pullback of $\mathcal{G}^{(n)}\to M$ along the usual jet projection $\pi^n{:}J^n(M, p)\to M$, which, assuming regularity, forms a subbundle $\mathcal{H}^{(n)}\subset\varepsilon^{(n)}$. Local coordinates on $\mathcal{H}^{(n)}$ are of the form $(x, u^{(n)}, \lambda^{(n)})$, where $(x, u^{(n)})$ are jet coordinates on $J^n(M, p)$ and the fiber coordinates $\lambda^{(n)}$ represent the pseudogroup parameters of order $\le n$.

Definition 1. An nth-order moving frame for a pseudogroup $\mathcal{G}$ acting on p-dimensional submanifolds $N\subset M$ is a locally $\mathcal{G}$-equivariant section $\rho^{(n)}{:}J^n(M, p)\to\mathcal{H}^{(n)}$.

The necessary and sufficient condition for the existence of a locally equivalent moving frame is given by the following theorem.

Theorem 1. A locally equivariant moving frame exists in a neighborhood of a jet $(x, u^{(n)})\in J^n(M, p)$ if and only if $\mathcal{G}$ acts locally freely at $(x, u^{(n)})$.

A practical way to construct a moving frame $\rho^{(n)}$ is through the normalization procedure based on the choice of a cross section to the $\mathcal{G}$-orbits. Once a moving frame is fixed, invariantizing the nth-order jet coordinates $(x, u^{(n)})$ leads to the normalized differential invariants,

$$H^i=\iota(x^i),\ I_J^\alpha=\iota(u_J^\alpha), \tag{7}$$

where ι is the induced invariantization process.

Theorem 2. Suppose the pseudogroup $\mathcal{G}$ admits a mutually compatible hierarchy of moving frames defined on suitable open subsets of $J^n(M, p)$ for $n\gg 0$. Then the nonphantom normalized differential invariants (7) of all orders $n\ge 0$ are functionally independent and generate the differential invariant algebra $I_{\mathcal{G}}$.

The more traditional way to get higher-order differential invariants is through invariant differential. A basis for the invariant differential operators $\mathcal{D}_1, \cdots$, and $\mathcal{D}_p$ can be obtained by invariantizing the total differential operators $D_1, \cdots$, and D_p. More explicitly, by invariantizing the horizontal coordinate coframe, we get the contact-invariant horizontal coframe

$$\omega^i=\iota(\mathrm{d}x^i),\ i=1, 2, \cdots, p. \tag{8}$$

To establish the recurrence formulas relating the normalized and differentiated invariants, the Maurer-Cartan forms for the diffeomorphism pseudogroup are necessary and essential, which are explicitly realized as the right-invariant contact forms on the infinite jet bundle. A basis is labeled by the fiber coordinates X_A^i, U_A^α, and

$$\chi_A^i,\mu_A^\alpha,\quad \text{for } i=1,\cdots,p,\quad \alpha=1,\cdots,q,\quad \#A\ge 0, \tag{9}$$

are used to denote the corresponding basis Maurer-Cartan forms.

Theorem 3. The restricted Maurer-Cartan forms satisfy the lifted determining equations

$$\mathcal{L}(\cdots,X^i,\cdots,U^\alpha,\cdots,\chi_A^i,\cdots,\mu_A^\alpha,\cdots)=0, \tag{10}$$

which are obtained by applying the following replacement rules:

$$x^i\mapsto X^i,\quad u^\alpha\mapsto U^\alpha,\quad \xi_A^i\mapsto\chi_A^i,\quad \varphi_A^\alpha\mapsto\mu_A^\alpha,$$

for all i, α, and A to the infinitesimal determining equations (6).

Nevertheless, in the construction of recurrence formulas, the most important form are not the Maurer-Cartan forms per se but their pullback under the moving frame map.

Definition 2. Given a moving frame $\rho^{(n)}{:}J^n(M, p)\to\mathcal{H}^{(n)}$, the invariantized Maurer-Cartan forms are defined as the horizontal components of the pullbacks

$$\beta_A^i=\pi_H[(\rho^{(n)})^*\chi_A^i],\quad \zeta_A^\alpha=\pi_H[(\rho^{(n)})^*\mu_A^\alpha]. \tag{11}$$

Theorem 4. The invariantized Maurer-Cartan forms satisfy the invariantized determining equations

$$\mathcal{L}(\cdots, H^i, \cdots, I^\alpha, \cdots, \beta^i_A, \cdots, \zeta^\alpha_A, \cdots)=0. \tag{12}$$

Extending the invariantization process, we set

$$\iota(\xi^i_A)=\beta^i_A, \quad \iota(\varphi^\alpha_A)=\zeta^\alpha_A, \tag{13}$$

to be the corresponding invariantized Maurer-Cartan forms (11).

Theorem 5. The recurrence formulas for the normalized differential invariants (7) are

$$\begin{aligned} \mathrm{d}_H H^j &= \sum_{i=1}^{p}(\mathcal{D}_i H^j)\omega^i=\omega^j+\beta^j, \\ \mathrm{d}_H I^\alpha_J &= \sum_{i=1}^{p}(\mathcal{D}_i I^\alpha_J)\omega^i=\sum_{i=1}^{p} I^\alpha_{J,i}\omega^i+\hat{\psi}^\alpha_J, \end{aligned} \tag{14}$$

in which $\hat{\psi}^\alpha_J=\iota(\hat{\varphi}^\alpha_J)$ is the invariantization of the coefficients of the prolonged vector field.

The general recurrence formula is as follows:

$$\mathcal{D}_K I^\alpha_J=I^\alpha_{J,K}+R^\alpha_{J,K}, \tag{15}$$

which is valid for any multi-indices J, K. In computations, the correction terms $R^\alpha_{J,K}$ are rewritten in terms of the generating differential invariants and their invariant derivatives.

The generating syzygies are of two types, the first one involves the syzygies of the form

$$\mathcal{D}_K I^\alpha_J=c^\alpha_{JK}+M^\alpha_{J,K}, \tag{16}$$

where I^α_J is a generating differential invariant and $I^\alpha_{JK}=c^\alpha_{JK}$ is a phantom differential invariant, whereas the second one consists of all equations of the form

$$\mathcal{D}_J I^\alpha_{LK}-\mathcal{D}_K I^\alpha_{LJ}=M^\alpha_{LK,J}-M^\alpha_{LJ,K}, \tag{17}$$

where I^α_{LK} and I^α_{LJ} are generating differential invariants, the multi-indices $K\cap J=\emptyset$ are disjoint and nonzero, and L is an arbitrary multi-index.

3 The Algebra of Differential Invariants

For the (2+1)-dimensional breaking soliton equation, the underlying total space is $M=\mathbb{R}^4$ with coordinates (t, x, y, u), and its solutions $u=f(t, x, y)$ define $p=3$-dimensional submanifolds of M. Its infinitesimal symmetry algebra consists of the vector fields,

$$\begin{aligned} \mathbf{v}=\tau(t, x, y, u)\frac{\partial}{\partial t}+\xi(t, x, y, u)\frac{\partial}{\partial x}+\eta(t, x, y, u)\frac{\partial}{\partial y} \\ +\varphi(t, x, y, u)\frac{\partial}{\partial u}, \end{aligned} \tag{18}$$

on M, and their prolongations,

$$\mathbf{v}^{(n)}=\tau\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\sum_{\#J\geq 0}\hat{\varphi}^J\frac{\partial}{\partial u_J}, \tag{19}$$

are tangent to the variety in $J^\infty(M, 3)$ defined by (1). This invariance condition leads to the infinitesimal determining equations along with their differential consequences reduce to the following system:

$$\begin{aligned} &\tau_x=\tau_y=\tau_u=\xi_y=\xi_u=\eta_x=\eta_u=0, \\ &\tau_{ttt}=\eta_{tt}=\eta_{yy}=0, \quad \xi_x=\frac{1}{2}(\tau_t-\eta_y), \quad \eta_{ty}=\frac{1}{2}\tau_{tt}, \\ &\varphi_x=-\frac{1}{4}\eta_t, \quad \varphi_y=-\frac{1}{2}\xi_t, \quad \varphi_u=\frac{1}{2}(\eta_y-\tau_t). \end{aligned} \tag{20}$$

Our actual choice of cross section, which defines the moving frame, will be deferred until we acquire some familiarity with the structure of the recurrence formulas. First, we take

$$H^1=\iota(t), \quad H^2=\iota(x), \quad H^3=\iota(y), \quad I_{ijk}=\iota(u_{ijk}),$$

to denote the corresponding normalized differential invariants and

$$\alpha_{ijkl}=\iota(\tau_{ijkl}), \quad \beta_{ijkl}=\iota(\xi_{ijkl}), \quad \gamma_{ijkl}=\iota(\eta_{ijkl}), \quad \zeta_{ijkl}=\iota(\varphi_{ijkl}),$$

to denote the invariantized Maurer-Cartan forms. The complete system of linear dependencies among the invariantized Maurer-Cartan forms can be obtained by the invariantization of the determining (20), namely,

$$\begin{aligned} &\alpha_X=\alpha_Y=\alpha_U=\beta_Y=\beta_U=\gamma_X=\gamma_U=0, \\ &\alpha_{TTT}=\gamma_{TT}=\gamma_{YY}=0, \quad \beta_X=\frac{1}{2}(\alpha_T-\gamma_Y), \quad \gamma_{TY}=\frac{1}{2}\alpha_{TT}, \\ &\zeta_X=-\frac{1}{4}\gamma_T, \quad \zeta_Y=-\frac{1}{2}\beta_T, \quad \zeta_U=\frac{1}{2}(\gamma_Y-\alpha_T), \end{aligned} \tag{21}$$

and so on. A basis of the invariantized Maurer-Cartan forms is provided as follows:

$$\alpha, \ \alpha_T, \ \alpha_{TT}, \ \gamma, \ \gamma_T, \ \gamma_Y, \ \beta_{T^n}, \ \zeta_{T^n}, \ n\geq 0.$$

Moreover, we let $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ be the invariant differential operators dual to the invariantized horizontal coframe

$$\omega^1=\iota(\mathrm{d}t), \quad \omega^2=\iota(\mathrm{d}x), \quad \omega^3=\iota(\mathrm{d}y). \tag{22}$$

As mentioned previously, the explicit formulas are not required at the moment.

The correction terms $\hat{\psi}_J^\alpha$ in (14) are the invariantization of the coefficients $\hat{\varphi}_J^\alpha$ of the prolonged vector field. Here without computing the explicit expressions of φ, only the prolongation formula of the vector field, the determining equations (20), and their differential consequences are needed to express $\hat{\varphi}_J^\alpha$.

Then (14) directly yield the following recurrence formulas:

$$\begin{aligned}
&\mathrm{d}_H H^1=\omega^1+\alpha,\quad \mathrm{d}_H H^2=\omega^2+\beta,\quad \mathrm{d}_H H^3=\omega^3+\gamma,\\
&\mathrm{d}_H I_{000}=I_{100}\omega^1+I_{010}\omega^2+I_{001}\omega^3+\zeta,\\
&\mathrm{d}_H I_{100}=I_{200}\omega^1+I_{110}\omega^2+I_{101}\omega^3+\frac{1}{2}I_{100}(\gamma_Y-3\alpha_T)-I_{010}\beta_T\\
&\qquad -I_{001}\gamma_T+\zeta_T,\\
&\mathrm{d}_H I_{010}=I_{110}\omega^1+I_{020}\omega^2+I_{011}\omega^3+I_{010}(\gamma_Y-\alpha_T)-\frac{1}{4}\gamma_T,\\
&\mathrm{d}_H I_{001}=I_{101}\omega^1+I_{011}\omega^2+I_{002}\omega^3-\frac{1}{2}I_{001}(\alpha_T+\gamma_Y)-\frac{1}{2}\beta_T,\\
&\mathrm{d}_H I_{110}=I_{210}\omega^1+I_{120}\omega^2+I_{111}\omega^3-\frac{1}{2}I_{010}\alpha_{TT}+I_{110}(\gamma_Y-2\alpha_T)\\
&\qquad -I_{020}\beta_T-I_{011}\gamma_T,\\
&\mathrm{d}_H I_{101}=I_{201}\omega^1+I_{111}\omega^2+I_{102}\omega^3-\frac{1}{2}\beta_{TT}-\frac{3}{4}I_{001}\alpha_{TT}\\
&\qquad -\frac{1}{2}I_{101}(3\alpha_T+\gamma_Y)-I_{011}\beta_T-I_{002}\gamma_T,\\
&\mathrm{d}_H I_{011}=I_{111}\omega^1+I_{021}\omega^2+I_{012}\omega^3-\frac{1}{8}\alpha_{TT}-I_{011}\alpha_T,\\
&\mathrm{d}_H I_{200}=I_{300}\omega^1+I_{210}\omega^2+I_{201}\omega^3+\frac{1}{2}I_{200}(\gamma_Y-5\alpha_T)\\
&\qquad -2I_{110}\beta_T-2I_{101}\gamma_T-\frac{3}{2}I_{100}\alpha_{TT}-I_{010}\beta_{TT}+\zeta_{TT},\\
&\mathrm{d}_H I_{020}=I_{120}\omega^1+I_{030}\omega^2+I_{021}\omega^3+\frac{3}{2}I_{020}(\gamma_Y-\alpha_T),\\
&\mathrm{d}_H I_{002}=I_{102}\omega^1+I_{012}\omega^2+I_{003}\omega^3-\frac{1}{2}I_{002}(\alpha_T+3\gamma_Y),\\
&\mathrm{d}_H I_{300}=I_{400}\omega^1+I_{310}\omega^2+I_{301}\omega^3+\frac{1}{2}I_{300}(\gamma_Y-7\alpha_T)-3I_{210}\beta_T\\
&\qquad -3I_{201}\gamma_T-\frac{15}{4}I_{200}\alpha_{TT}-3I_{110}\beta_{TT}-I_{010}\beta_{TTT}+\zeta_{TTT},\\
&\mathrm{d}_H I_{030}=I_{130}\omega^1+I_{040}\omega^2+I_{031}\omega^3+2I_{030}(\gamma_Y-\alpha_T),\\
&\mathrm{d}_H I_{003}=I_{103}\omega^1+I_{013}\omega^2+I_{004}\omega^3-\frac{1}{2}I_{003}(\alpha_T+5\gamma_Y),\\
&\mathrm{d}_H I_{120}=I_{220}\omega^1+I_{130}\omega^2+I_{121}\omega^3+\frac{1}{2}I_{120}(3\gamma_Y-5\alpha_T)\\
&\qquad -I_{030}\beta_T-I_{021}\gamma_T-\frac{3}{4}I_{020}\alpha_{TT},\\
&\mathrm{d}_H I_{210}=I_{310}\omega^1+I_{220}\omega^2+I_{211}\omega^3+I_{210}(\gamma_Y-3\alpha_T)-2I_{120}\beta_T\\
&\qquad -2I_{111}\gamma_T-2I_{110}\alpha_{TT}-I_{020}\beta_{TT},\\
&\mathrm{d}_H I_{102}=I_{202}\omega^1+I_{112}\omega^2+I_{103}\omega^3-\frac{3}{2}I_{102}(\alpha_T+\gamma_Y)-I_{012}\beta_T\\
&\qquad -I_{003}\gamma_T-\frac{5}{4}I_{002}\alpha_{TT},\\
&\mathrm{d}_H I_{201}=I_{301}\omega^1+I_{211}\omega^2+I_{202}\omega^3-\frac{1}{2}I_{201}(5\alpha_T+\gamma_Y)-2I_{111}\beta_T\\
&\qquad -2I_{102}\gamma_T-\frac{5}{2}I_{101}\alpha_{TT}-I_{011}\beta_{TT}-\frac{1}{2}\beta_{TTT},\\
&\mathrm{d}_H I_{012}=I_{112}\omega^1+I_{022}\omega^2+I_{013}\omega^3-I_{012}(\alpha_T+\gamma_Y),\\
&\mathrm{d}_H I_{021}=I_{121}\omega^1+I_{031}\omega^2+I_{022}\omega^3+\frac{1}{2}I_{021}(\gamma_Y-3\alpha_T),\\
&\vdots
\end{aligned}\tag{23}$$

Here we choose the following normalizations:

$$\begin{aligned}
&H^1\mapsto 0,\quad H^2\mapsto 0,\quad H^3\mapsto 0,\quad I_{000}\mapsto 0,\quad I_{100}\mapsto 0,\quad I_{010}\mapsto 0,\\
&I_{001}\mapsto 0,\quad I_{011}\mapsto 0,\quad I_{101}\mapsto 0,\quad I_{200}\mapsto 0,\quad I_{020}\mapsto 1,\quad I_{002}\mapsto 1,\\
&I_{i,0,0}\mapsto 0,\quad I_{i-1,0,1}\mapsto 0,\quad \text{for all } i\ge 3,
\end{aligned}\tag{24}$$

which, when substituted into (23), yield the following expressions:

$$\begin{aligned}
&\alpha=-\omega^1,\quad \beta=-\omega^2,\quad \gamma=-\omega^3,\quad \zeta=0,\\
&\alpha_T=\frac{1}{2}(I_{102}+I_{120})\omega^1+\frac{1}{2}(I_{012}+I_{030})\omega^2+\frac{1}{2}(I_{021}+I_{003})\omega^3,\\
&\alpha_{TT}=8I_{111}\omega^1+8I_{021}\omega^2+8I_{012}\omega^3,\\
&\beta_T=2\omega^3,\quad \beta_{TT}=-8I_{110}\omega^1+2(I_{111}-4)\omega^2+2I_{102}\omega^3,\\
&\beta_{TTT}=2(I_{301}-8I_{110}I_{102})\omega^1+2(I_{211}-8I_{102})\omega^2\\
&\qquad +2(I_{202}-4I_{111})\omega^3,\quad \cdots,\\
&\gamma_T=4I_{110}\omega^1+4\omega^2,\quad \gamma_Y=\frac{1}{2}\left(I_{102}-\frac{1}{3}I_{120}\right)\omega^1\\
&\qquad +\frac{1}{2}\left(I_{012}-\frac{1}{3}I_{030}\right)\omega^2+\frac{1}{2}\left(I_{003}-\frac{1}{3}I_{021}\right)\omega^3,\\
&\zeta_T=-I_{110}\omega^2,\quad \zeta_{TT}=-I_{210}\omega^2+4I_{110}\omega^3,\\
&\zeta_{TTT}=-(24I_{110}^2+I_{400})\omega^1+(6I_{110}I_{111}-24I_{110}-I_{310})\omega^2\\
&\qquad +(6I_{110}I_{102}+6I_{210}-I_{301})\omega^3,\quad \cdots,
\end{aligned}\tag{25}$$

for the basic invariant forms. However, the higher-order invariantized Maurer-Cartan forms can be recursively determined from them.

Next, (25) was substituted into the equations for the nonphantom variables in (23) to derive the recurrence formulas between the differentiated and the normalized invariants:

$$\mathcal{D}_1 I_{110}=I_{210}-\frac{1}{2}I_{110}\left(I_{102}+\frac{7}{3}I_{120}\right),$$

$$\mathcal{D}_2 I_{110}=I_{120}-\frac{1}{2}I_{110}\left(I_{012}+\frac{7}{3}I_{030}\right),$$

$$\mathcal{D}_3 I_{110} = I_{111} - \frac{1}{2} I_{110}\left(I_{003} + \frac{7}{3} I_{021}\right) - 2,$$

$$\mathcal{D}_1 I_{030} = I_{130} - \frac{4}{3} I_{030} I_{120},$$

$$\mathcal{D}_2 I_{030} = I_{040} - \frac{4}{3} I_{030}^2,$$

$$\mathcal{D}_3 I_{030} = I_{031} - \frac{4}{3} I_{030} I_{021},$$

$$\mathcal{D}_1 I_{003} = I_{103} - \frac{1}{2} I_{003}\left(3I_{102} - \frac{1}{3} I_{120}\right),$$

$$\mathcal{D}_2 I_{003} = I_{013} - \frac{1}{2} I_{003}\left(3I_{012} - \frac{1}{3} I_{030}\right),$$

$$\mathcal{D}_3 I_{003} = I_{004} - \frac{1}{2} I_{003}\left(3I_{003} - \frac{1}{3} I_{021}\right),$$

$$\mathcal{D}_1 I_{120} = I_{220} - \frac{1}{2} I_{120}(I_{102} + 3I_{120}) - 4I_{110} I_{021} - 6I_{111},$$

$$\mathcal{D}_2 I_{120} = I_{130} - \frac{1}{2} I_{120}(I_{012} + 3I_{030}) - 10I_{021},$$

$$\mathcal{D}_3 I_{120} = I_{121} - \frac{1}{2} I_{120}(I_{003} + 3I_{021}) - 6I_{012} - 2I_{030},$$

$$\mathcal{D}_1 I_{210} = I_{310} - I_{210}\left(I_{102} + \frac{5}{3} I_{120}\right) - 8I_{110}(3I_{111} - 1),$$

$$\mathcal{D}_2 I_{210} = I_{220} - I_{210}\left(I_{012} + \frac{5}{3} I_{030}\right) - 16I_{110} I_{021} - 10I_{111} + 8,$$

$$\mathcal{D}_3 I_{210} = I_{211} - I_{210}\left(I_{003} + \frac{5}{3} I_{021}\right) - 16I_{110} I_{012} - 2I_{102} - 4I_{120},$$

$$\mathcal{D}_1 I_{102} = I_{202} - \frac{1}{2} I_{102}(I_{120} + 3I_{102}) - 4I_{110} I_{003} - 10I_{111},$$

$$\mathcal{D}_2 I_{102} = I_{112} - \frac{1}{2} I_{102}(I_{030} + 3I_{012}) - 4I_{003} - 10I_{021},$$

$$\mathcal{D}_3 I_{102} = I_{103} - \frac{1}{2} I_{102}(I_{021} + 3I_{003}) - 12I_{012},$$

$$\mathcal{D}_1 I_{012} = I_{112} - I_{012}\left(I_{102} + \frac{1}{3} I_{120}\right),$$

$$\mathcal{D}_2 I_{012} = I_{022} - I_{012}\left(I_{012} + \frac{1}{3} I_{030}\right),$$

$$\mathcal{D}_3 I_{012} = I_{013} - I_{012}\left(I_{003} + \frac{1}{3} I_{021}\right),$$

$$\mathcal{D}_1 I_{021} = I_{121} - \frac{1}{2} I_{021}\left(I_{102} + \frac{5}{3} I_{120}\right),$$

$$\mathcal{D}_2 I_{021} = I_{031} - \frac{1}{2} I_{021}\left(I_{012} + \frac{5}{3} I_{030}\right),$$

$$\mathcal{D}_3 I_{021} = I_{022} - \frac{1}{2} I_{021}\left(I_{003} + \frac{5}{3} I_{021}\right),$$

$$\vdots \tag{26}$$

Following the method of Olver and Pohjanpelto [14], we calculate that the invariant horizontal one-forms ω^1, ω^2, and ω^3 satisfy the following structure equations:

$$\begin{aligned}
\mathrm{d}_H \omega^1 &= -\frac{1}{2}(I_{012} + I_{030})\omega^1 \wedge \omega^2 - \frac{1}{2}(I_{003} + I_{021})\omega^1 \wedge \omega^3,\\
\mathrm{d}_H \omega^2 &= \frac{1}{3}\omega^1 \wedge \omega^2 - 2\omega^1 \wedge \omega^3 - \frac{1}{3} I_{021}\omega^2 \wedge \omega^3,\\
\mathrm{d}_H \omega^3 &= -4\omega^1 \wedge \omega^2 + \frac{1}{2}\left(I_{102} - \frac{1}{3} I_{120}\right)\omega^1 \wedge \omega^3\\
&\quad + \frac{1}{2}\left(I_{012} - \frac{1}{3} I_{030}\right)\omega^2 \wedge \omega^3.
\end{aligned} \tag{27}$$

By duality, (27) implies the commutation relations among the invariant differential operators $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$,

$$\begin{aligned}
[\mathcal{D}_1, \mathcal{D}_2] &= \frac{1}{2}(I_{012} + I_{030})\mathcal{D}_1 - \frac{1}{3}\mathcal{D}_2 + 4\mathcal{D}_3,\\
[\mathcal{D}_1, \mathcal{D}_3] &= \frac{1}{2}(I_{003} + I_{021})\mathcal{D}_1 + 2\mathcal{D}_2 - \frac{1}{2}\left(I_{102} - \frac{1}{3} I_{120}\right)\mathcal{D}_3,\\
[\mathcal{D}_2, \mathcal{D}_3] &= \frac{1}{3} I_{021}\mathcal{D}_2 - \frac{1}{2}\left(I_{012} - \frac{1}{3} I_{030}\right)\mathcal{D}_3.
\end{aligned} \tag{28}$$

Then the higher-order normalized invariants can be obtained in terms of the lower-order invariants by the repeated application of the recurrence formulas (26). Specifically, we can express I_{210} in terms of I_{110}, I_{012}, I_{102}, and I_{030} and the invariant derivative of I_{110} from the first two equations (26). Furthermore, the following fundamental syzygies among the basic differential invariants I_{110}, I_{102}, I_{012}, I_{021}, I_{030}, and I_{003} are derived as follows:

$$\mathcal{D}_3 I_{030} - \mathcal{D}_2 I_{021} + \frac{1}{2} I_{021}(I_{012} + I_{030}) = 0,$$

$$\mathcal{D}_2 I_{003} - \mathcal{D}_3 I_{012} + \frac{1}{2} I_{003}\left(I_{012} - \frac{1}{3} I_{030}\right) - \frac{1}{3} I_{012} I_{021} = 0,$$

$$\mathcal{D}_2 I_{012} - \mathcal{D}_3 I_{021} + I_{012}\left(I_{012} + \frac{1}{3} I_{030}\right) - \frac{1}{2} I_{021}\left(I_{003} + \frac{5}{3} I_{021}\right) = 0,$$

$$\begin{aligned}
&\mathcal{D}_1 I_{003} - \mathcal{D}_3 I_{102} - \frac{1}{6}\left(\mathcal{D}_2 I_{110} + \frac{1}{6}(3I_{012} + 7I_{030})I_{110}\right)I_{003}\\
&\quad - \frac{1}{2}(I_{021} + 24)I_{012} = 0,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{D}_2\mathcal{D}_3 I_{110} - \mathcal{D}_1 I_{021} + \frac{1}{2}(I_{012} + I_{030})I_{003} + \frac{1}{4}\left(3I_{012} + \frac{7}{9} I_{030}\right)I_{021}\\
&\quad + \frac{1}{2}\mathcal{D}_2 I_{003} + \frac{7}{6}\mathcal{D}_3 I_{021} = 0,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{D}_2 I_{102} - \mathcal{D}_1 I_{012} + \frac{1}{2}(I_{012} + I_{030})I_{102}\\
&\quad - \frac{1}{3}\left(\mathcal{D}_2 I_{110} + \frac{1}{6}(3I_{012} + 7I_{030})I_{110}\right)I_{012} + 4I_{003} + 10I_{021} = 0,
\end{aligned}$$

$$
\begin{aligned}
&\mathcal{D}_2^2 I_{110} - \mathcal{D}_1 I_{030} + \left(\left(\frac{1}{4} I_{012} + \frac{2}{3} I_{030}\right) I_{012}\right.\\
&\quad + \frac{7}{6}\left(\mathcal{D}_2 I_{030} + \frac{1}{6} I_{030}^2\right) + \frac{1}{2}\mathcal{D}_2 I_{012}\bigg) I_{110} + \left(I_{012} + \frac{4}{3} I_{030}\right)\mathcal{D}_2 I_{110}\\
&\quad + 10 I_{021} = 0,\\
&\mathcal{D}_3^2 I_{110} - \mathcal{D}_2 I_{102} + \left(\frac{1}{2}\mathcal{D}_3 I_{003} + \left(\frac{1}{2} I_{003} + \frac{5}{3} I_{021}\right) I_{003}\right.\\
&\quad + \frac{7}{6}(I_{021}^2 + \mathcal{D}_3 I_{021})\bigg) I_{110} + \frac{1}{2}\left(3 I_{003} + \frac{13}{3} I_{021}\right)\mathcal{D}_3 I_{110}\\
&\quad - 2 I_{003} - 6 I_{021} - \frac{3}{2} I_{012} - \frac{1}{2} I_{030} = 0,\\
&\mathcal{D}_3^2 I_{110} - \mathcal{D}_1 I_{012} + \left(\frac{1}{3}\mathcal{D}_2 I_{110} + I_{102} - \frac{1}{6}\left(I_{012} - \frac{7}{3} I_{030}\right) I_{110}\right) I_{012}\\
&\quad + 4 I_{021} + 2 I_{003} + \left(\frac{1}{2}\mathcal{D}_3 I_{003} + \left(\frac{1}{2} I_{003} + \frac{5}{3} I_{021}\right) I_{003}\right.\\
&\quad + \frac{7}{6}(\mathcal{D}_3 I_{021} + I_{021}^2)\bigg) I_{110} + \frac{1}{2}\left(3 I_{003} + \frac{13}{3} I_{021}\right)\mathcal{D}_3 I_{110} = 0. \quad (29)
\end{aligned}
$$

Theorem 6. The differential invariants $I_{110}, I_{102}, I_{012}, I_{021}, I_{030}$, and I_{003} form a generating set for the algebra of differential invariants for the symmetry pseudogroup of the breaking soliton equation.

The recurrence formulas (26) and the generating syzygies (29), along with the commutation relations (28), serve to completely specify the structure of the differential invariant algebra for the breaking soliton equation. Remember that so far we have only used the infinitesimal determining equations and choice of cross-sectional normalization to completely determine this intricate structure. In the next part, the explicit formulas for the symmetry pseudogroup transformations are constructed to derive explicit formulas for the moving frame, the differential invariants, and the invariant differential operators.

4 A Moving Frame and Invariantization

The solution to the infinitesimal determining (20) is given by

$$
\begin{aligned}
&\tau = \frac{1}{2}\lambda_1 t^2 + \lambda_2 t + \lambda_3, \quad \xi = \frac{1}{4}(\lambda_1 t + 2\lambda_2 - 2\lambda_5)x + F,\\
&\eta = \frac{1}{2}(\lambda_1 y + 2\lambda_4)t + \lambda_5 y + \lambda_6,\\
&\varphi = -\frac{1}{4}(\lambda_1 t + 2\lambda_2 - 2\lambda_5)u - \frac{1}{4}\lambda_4 x - \frac{1}{8}\lambda_1 xy - \frac{1}{2}yF' + G, \qquad (30)
\end{aligned}
$$

where λ_i, $i = 1, 2, \cdots, 6$ are constants, $F \equiv F(t)$, and $G \equiv G(t)$ are arbitrary smooth functions. The prime is used to denote derivative. According to the standard algorithm for constructing a group action from the infinitesimal generators, we get the following explicit transformations:

$$
\begin{aligned}
&T = \frac{2(t + \lambda_3)}{2 - \lambda_1(t + \lambda_3)} e^{\lambda_2},\\
&X = \frac{\sqrt{2}(x + F)}{(2 - \lambda_1(t + \lambda_3))^{1/2}} e^{\frac{1}{2}(\lambda_2 - \lambda_5)},\\
&Y = \frac{2(\lambda_4 t + y + \lambda_6)}{2 - \lambda_1(t + \lambda_3)} e^{\lambda_5},\\
&U = \frac{\sqrt{2}}{8}\frac{4(2 - \lambda_1(t + \lambda_3))K - \lambda_1(\lambda_4 t + y + \lambda_6)(x + F)}{(2 - \lambda_1(t + \lambda_3))^{1/2}} e^{\frac{1}{2}(\lambda_5 - \lambda_2)},
\end{aligned}
\qquad (31)
$$

where $K = u - \frac{1}{2}yF' - \frac{\lambda_4}{4}(x + F) + G$.

The lifted horizontal coframe associated to the pseudogroup action (31) is given by

$$
\begin{aligned}
&\mathrm{d}_H T = \frac{4}{(2 - \lambda_1(t + \lambda_3))^2} e^{\lambda_2}\mathrm{d}t,\\
&\mathrm{d}_H X = \frac{\sqrt{2}}{(2 - \lambda_1(t + \lambda_3))^{1/2}} e^{\frac{1}{2}(\lambda_2 - \lambda_5)}\left(\left(F' + \frac{\lambda_1(x + F)}{2(2 - \lambda_1(t + \lambda_3))}\right)\mathrm{d}t + \mathrm{d}x\right),\\
&\mathrm{d}_H Y = \frac{2}{2 - \lambda_1(t + \lambda_3)} e^{\lambda_5}((\lambda_1(y + \lambda_6) + \lambda_4(2 - \lambda_1\lambda_3))\mathrm{d}t - \mathrm{d}y), \qquad (32)
\end{aligned}
$$

with dual lifted total differential operators

$$
\begin{aligned}
&D_T = \frac{1}{4}e^{-\lambda_2}(2 - \lambda_1(t + \lambda_3))^2 D_t - \frac{1}{4}e^{-\lambda_2}(2 - \lambda_1(t + \lambda_3))\\
&\left(\left(\frac{\lambda_1}{2}(x + F) + (2 - \lambda_1(t + \lambda_3))F'\right)D_x + (\lambda_1(y + \lambda_6 - \lambda_3\lambda_4) + 2\lambda_4)D_y\right),\\
&D_X = \frac{1}{\sqrt{2}} e^{\frac{1}{2}(\lambda_5 - \lambda_2)}(2 - \lambda_1(t + \lambda_3))^{1/2} D_x,\\
&D_Y = \frac{1}{2}e^{-\lambda_5}(2 - \lambda_1(t + \lambda_3))D_y. \qquad (33)
\end{aligned}
$$

Then the prolonged pseudogroup action of the symmetry algebra on submanifold jets can be obtained by repeatedly applying the differential operators in (33) to U in (31). For instance,

$$
\begin{aligned}
\hat{U}_X &= \frac{1}{2}e^{\lambda_5-\lambda_2}\left((2-\lambda_1(t+\lambda_3))u_x - \frac{1}{4}(\lambda_1(y+\lambda_6-\lambda_3\lambda_4)+2\lambda_4)\right),\\
\hat{U}_Y &= \frac{\sqrt{2}}{4}e^{\frac{1}{2}(\lambda_2+\lambda_5)}(2-\lambda_1(t+\lambda_3))^{1/2}\\
&\quad\left((2-\lambda_1(t+\lambda_3))\left(u_y-\frac{1}{2}F'\right)-\frac{\lambda_1}{4}(x+F)\right),\\
\hat{U}_{XX} &= \frac{\sqrt{2}}{4}e^{\frac{3}{2}(\lambda_5-\lambda_2)}(2-\lambda_1(t+\lambda_3))^{3/2}u_{xx},\\
\hat{U}_{XY} &= \frac{1}{4}e^{-\lambda_2}(2-\lambda_1(t+\lambda_3))\left((2-\lambda_1(t+\lambda_3))u_{xy}-\frac{\lambda_1}{4}\right),\\
\hat{U}_{YY} &= \frac{\sqrt{2}}{8}e^{-\frac{1}{2}(\lambda_2+3\lambda_5)}(2-\lambda_1(t+\lambda_3))^{5/2}u_{yy},\\
\hat{U}_{XXY} &= \frac{\sqrt{2}}{8}e^{\frac{1}{2}(\lambda_5-3\lambda_2)}(2-\lambda_1(t+\lambda_3))^{5/2}u_{xxy},\\
\hat{U}_{XXXY} &= \frac{1}{8}e^{\lambda_5-2\lambda_2}(2-\lambda_1(t+\lambda_3))^{3}u_{xxxy},\quad \cdots.
\end{aligned}\tag{34}
$$

On the subset $\mathcal{V}=\{u_{xx}u_{yy}>0\}$, we can solve the normalization equations (24) for the pseudogroup parameters

$$
\begin{aligned}
&\lambda_1 = 8u_{xy},\quad \lambda_2=\frac{1}{2}\ln(u_{xx}u_{yy}),\quad \lambda_3=-t,\\
&\lambda_4 = 4u_x,\quad \lambda_5=\frac{1}{2}\ln(u_{xx}^{-1/3}u_{yy}),\quad \lambda_6=-y-4tu_x,\\
&F=-x,\quad F'=2u_y\quad F''=2u_{ty}-8u_xu_{yy}-4u_{xy}u_y,\quad \cdots,\\
&G=yu_y-u,\quad G'=y(u_{ty}-4u_xu_{yy})+2u_y(u_x-yu_{xy})-u_t,\quad \cdots.
\end{aligned}\tag{35}
$$

Substituting (35) into the jet coordinates of transformed submanifold yields the normalized differential invariants

$$
\begin{aligned}
&I_{000}=\iota(u)=0,\quad I_{100}=\iota(u_t)=0,\\
&I_{010}=\iota(u_x)=0,\quad I_{001}=\iota(u_y)=0,\\
&I_{110}=\iota(u_{tx})=(u_{tx}-4u_xu_{xy}-2u_{xx}u_y)u_{xx}^{-7/6}u_{yy}^{-1/2},\\
&I_{101}=\iota(u_{ty})=0,\quad I_{020}=\iota(u_{xx})=1,\\
&I_{002}=\iota(u_{yy})=1,\quad I_{011}=\iota(u_{xy})=0,\\
&I_{031}=\iota(u_{xxxy})=u_{xxxy}u_{xx}^{-7/6}u_{yy}^{-1/2},\quad \cdots.
\end{aligned}\tag{36}
$$

By substituting (35) into (33), we get the invariant differential operators

$$
\begin{aligned}
&\mathcal{D}_1=(u_{xx}u_{yy})^{-1/2}(D_t-2u_yD_x-4u_xD_y),\\
&\mathcal{D}_2=u_{xx}^{-1/3}D_x,\quad \mathcal{D}_3=u_{xx}^{1/6}u_{yy}^{-1/2}D_y.
\end{aligned}\tag{37}
$$

Finally, the (2+1)-dimensional breaking soliton equation (1) can be immediately rewritten in terms of the differential invariants obtained by applying the induced invariantization process ι to it:

$$
\begin{aligned}
0&=\iota(u_{tx}-4u_xu_{xy}-2u_{xx}u_y-u_{xxxy})\\
&=I_{110}-I_{031}=(u_{tx}-4u_xu_{xy}-2u_{xx}u_y-u_{xxxy})u_{xx}^{-7/6}u_{yy}^{-1/2}.
\end{aligned}\tag{38}
$$

5 Conclusions

The equivariant moving frame method has been proven to be a very powerful tool in studying the differential invariants for Lie pseudogroups. In this paper, only using the infinitesimal determining equations and choosing of cross-sectional normalization, we have completely determined the algebra structure of differential invariant for the breaking soliton equation. The complete classification of the differential invariants, their syzygies, and recurrence relations are obtained. These results are useful in performing a wide variety of applications of differential invariants of the symmetry pseudogroup for the breaking soliton equation. Furthermore, how to solve the original equation via the differential invariants obtained, the relationship among Lax pairs, the Maurer-Cartan forms, and the differential invariant algebra, which are interesting and meaningful, deserves our further research.

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