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Symmetry Analysis and Exact Solutions of the 2D Unsteady Incompressible Boundary-Layer Equations*

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Abstract *To find intrinsically different symmetry reductions and inequivalent group invariant solutions of the 2D unsteady incompressible boundary-layer equations, a two-dimensional optimal system is constructed which attributed to the classification of the corresponding Lie subalgebras. The comprehensiveness and inequivalence of the optimal system are shown clearly under different values of invariants. Then by virtue of the optimal system obtained, the boundary-layer equations are directly reduced to a system of ordinary differential equations (ODEs) by only one step. It has been shown that not only do we recover many of the known results but also find some new reductions and explicit solutions, which may be previously unknown.*

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1 Introduction

Group theoretic methods are useful to study similarity reductions and exact solutions of partial differential equations (PDEs).[1−4] The classical symmetry methods due to Lie have been generalized to the nonclassical case by Bluman and Cole.[5] Generally, an s-parameter subgroup of the full symmetry group of a system of PDEs with $n > s$ independent variables leads to a family of group invariant solutions. Unfortunately, it is usually not practicable to list all possible group invariant solutions of the system as there exist almost an infinite number of subgroups. Therefore, to construct all the inequivalent group invariant solutions is anticipated, or equivalently, to give them a classification, which leads to the concept of optimal system. In practical, an optimal system of the Lie algebra is constructed, from which the corresponding optimal systems for group invariant solutions are obtained. The adjoint representation of a Lie group on its Lie algebra is known to Lie and its application in classifying group invariant solutions appears from Ovsiannikov.[2] By using a global matrix for the adjoint transformation, Ovsiannikov demonstrates the construction of a one-dimensional optimal system and sketches the construction of higher-dimensional optimal systems. Olver adopts a slightly different and elegant way by taking a table of adjoint operators to simplify a general element from the Lie algebra as much as possible.[3] Following Olver's method, we have obtained the optimal systems as well as some interesting exact solutions for several important PDEs appeared in hydrodynamics and atmosphere.[6−8] Some other works on optimal systems can also be found in Refs. [9–11]. Very recently, a direct algorithm of one-dimensional and two-dimensional optimal system is introduced in Refs. [12] and [23], respectively. The algorithm can guarantee both comprehensiveness and inequivalence of the elements in the optimal system obtained, need no further proofs.

As a matter of fact, the two-dimensional unsteady flow of an incompressible viscous fluid is governed by the well-know Navier–Stokes equations

$$u_x + v_y = 0\,, \quad u_t + uu_x + vu_y = -p_x + \nu(u_{xx} + u_{yy})\,,$$
$$v_t + uv_x + vv_y = -p_y + \nu(v_{xx} + v_{yy})\,, \tag{1}$$

where $u \equiv u(x, y, t)$ and $v \equiv v(x, y, t)$ are velocity components in the x- and y-directions, respectively. $p \equiv p(x, y, t)$ is fluid pressure and ν is kinematic viscosity. Here and further the subscripts denote partial differentiation with respect to corresponding variables. For simplicity, we can set $\nu = 1$ by choosing suitable units for length and time.

About a century ago, Prandtl found that boundary layers play an important role in determining precisely the flow of certain fluids.[14] He demonstrated that for slightly viscous flows the viscosity plays a key role near boundaries, although it can be negligible in the most of the flow. For an incompressible two-dimensional viscous fluid over a flat plate with the latter taken as $y = 0$, the unsteady laminar boundary-layer equations for flow of high Reynolds number are given by

$$u_x + v_y = 0\,,$$
$$u_t + uu_x + vu_y = w_t + ww_x + u_{yy}\,,$$

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$$w_y = 0\,. \tag{2}$$

In addition, for the stationary flat plate, the boundary conditions can be taken as

$$u(x,0,t) = v(x,0,t) = 0\,,$$
$$u(x,y,t) = w(x,t) \quad \text{as} \quad y \to \infty\,. \tag{3}$$

The external inviscid flow $w(x,t)$ is related to the pressure p through

$$w_t + w w_x = -p_x\,, \tag{4}$$

and it is usually obtained via inviscid flow calculations. Besides the boundary conditions, an initial value of u is needed to form a well-posed problem. However, in studying similarity solutions, the initial condition is determined rather than prescribed.

The quest for similarity reductions and exact solutions of the boundary-layer equations is significant and also has a long history.[2,14–21] The motivation of the current paper is to perform symmetry analysis of the boundary-layer equations (2) along with the boundary conditions (3). A Lie subalgebra of the full symmetry group of the boundary-layer equations which also leaves the boundary conditions (3) invariant is considered. To get intrinsically different symmetry reductions and inequivalent group invariant solutions, a two-dimensional optimal system of the subalgebra is constructed. It is necessary to point that similarity reductions have many important values other than as mere mathematical exercises. Actually, reductions and exact solutions of physically important PDEs (including the boundary-layer equations) are significant. Some exact solutions describe an important physical phenomenon and solutions of a system asymptotically tend to the solutions of the corresponding lower dimensional system obtained through similarity reductions. What is more, exact solutions also provide valuable checks on the accuracy and reliability of numerical algorithms.

The layout of this paper is as follows: In Sec. 2, symmetry analysis and a two-dimensional optimal system of the boundary-layer equations (2) and (3) are presented. Then with aid of the optimal system obtained, symmetry reductions and exact solutions of the boundary-layer equations are performed in Sec. 3. The last section is a short conclusion.

2 Construction of a Two-Dimensional Optimal System

2.1 *Lie Algebra of the Boundary-Layer Equations*

To perform Lie symmetry analysis of Eqs. (2), we consider the one-parameter Lie group of infinitesimal transformations in the form

$$\tilde{x} = x + \epsilon\xi^x(x,y,t,u,v,w) + \mathcal{O}(\epsilon^2)\,,$$
$$\tilde{y} = y + \epsilon\xi^y(x,y,t,u,v,w) + \mathcal{O}(\epsilon^2)\,,$$
$$\tilde{t} = t + \epsilon\xi^t(x,y,t,u,v,w) + \mathcal{O}(\epsilon^2)\,,$$
$$\tilde{u} = u + \epsilon\eta^u(x,y,t,u,v,w) + \mathcal{O}(\epsilon^2)\,,$$
$$\tilde{v} = v + \epsilon\eta^v(x,y,t,u,v,w) + \mathcal{O}(\epsilon^2)\,,$$
$$\tilde{w} = w + \epsilon\eta^w(x,y,t,u,v,w) + \mathcal{O}(\epsilon^2)\,, \tag{5}$$

with $\epsilon \ll 1$ is the group parameter. It is required that the set of solutions of Eqs. (2) is invariant under the transformation (5), which leads to a system of over-determined, linear equations for the infinitesimals ξ^x, ξ^y, ξ^t, η^u, η^v, and η^w. The associated Lie algebra of infinitesimal transformations is spanned by the set of vector fields

$$v = \xi^x\frac{\partial}{\partial x} + \xi^y\frac{\partial}{\partial y} + \xi^t\frac{\partial}{\partial t} + \eta^u\frac{\partial}{\partial u} + \eta^v\frac{\partial}{\partial v} + \eta^w\frac{\partial}{\partial w}\,. \tag{6}$$

Here, the vector fields (6) have no relationship to the velocity variable v. For Eqs. (2), the solution of the determining equations is given by

$$\xi^t = c_1 t + c_2\,, \quad \xi^x = c_3 x + r\,, \quad \xi^y = \frac{1}{2}c_1 y + s\,,$$
$$\eta^u = (c_3 - c_1)u + r'\,, \quad \eta^v = -\frac{1}{2}c_1 v + s_t + s_x u\,,$$
$$\eta^w = (c_3 - c_1)w + r'\,, \tag{7}$$

where c_i, $i = 1,2,3$ are constants, $r \equiv r(t)$ and $s \equiv s(x,t)$ are arbitrary smooth functions, and the prime denotes derivative with respect to time t. So the Lie algebra of the symmetry group of Eqs. (2) is spanned by the following vector fields

$$v_1 = \frac{\partial}{\partial t}\,, \quad v_2 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + w\frac{\partial}{\partial w}\,,$$
$$v_3 = t\frac{\partial}{\partial t} + \frac{1}{2}y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - \frac{1}{2}v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w}\,,$$
$$v_r = r\frac{\partial}{\partial x} + r'\frac{\partial}{\partial u} + r'\frac{\partial}{\partial w}\,,$$
$$v_s = s\frac{\partial}{\partial y} + (s_t + s_x u)\frac{\partial}{\partial v}\,. \tag{8}$$

Remark 1 Comparing with Ovsiannikov's solution,[2] one can see that his solution is not the most general solution as it loses the x dependence of the function s. Ma and Hui also get the Lie symmetry algebra of Eqs. (2).[20] Their solution is slightly different from our's, as the calculation of the optimal system is simpler in terms of Eq. (8), so we consider (8) in this paper.

To leave the boundary conditions (3) invariant,[4] we must take r = constant and $s = 0$. Hence, in the subsection, a subalgebra of Eq. (8) spanned by the following four vector fields will be considered

$$v_1 = \frac{\partial}{\partial t}\,, \quad v_2 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + w\frac{\partial}{\partial w}\,,$$
$$v_3 = t\frac{\partial}{\partial t} + \frac{1}{2}y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - \frac{1}{2}v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w}\,,$$
$$v_4 = \frac{\partial}{\partial x}\,. \tag{9}$$

The commutation relations between the vector fields v_i and v_j in Eq. (9) are given in Table 1. In which, the entry in row i and column j representing $[v_i, v_j] = v_i v_j - v_j v_i$.

To compute the adjoint representation, we use the Lie series in conjunction with Table 1. Applying the formula

$$Ad_{\exp(\epsilon v_i)}v_j = v_j - \epsilon[v_i, v_j] + \frac{1}{2}\epsilon^2[v_i,[v_i,v_j]] - \cdots, \quad (10)$$

one can construct the following adjoint representation table with the (i,j)-th entry indicating $Ad_{\exp(\epsilon v_i)}v_j$.

Table 1 Commutator table of the Lie algebra (9).

$[v_i, v_j]$	v_1	v_3	v_3	v_4
v_1	0	0	v_1	0
v_2	0	0	0	$-v_4$
v_3	$-v_1$	0	0	0
v_4	0	v_4	0	0

In conjunction with Table 2, the general adjoint transformation matrix A can be obtained, which is the product of the matrices of the separate adjoint actions A_1, A_2, A_3, A_4 in any order and it is useful in the construction of an optimal system. The orders of the product of A_1, A_2, A_3, A_4 are not important since only the existence of the elements of the group is needed in the algorithm.

Table 2 Adjoint representation table of the Lie algebra (9).

$Ad_{\exp(\epsilon v_i)}v_j$	v_1	v_2	v_3	v_4
v_1	v_1	v_2	$v_3 - \epsilon v_1$	v_4
v_2	v_1	v_2	v_3	$e^{\epsilon}v_4$
v_3	$e^{\epsilon}v_1$	v_2	v_3	v_4
v_4	v_1	$v_2 - \epsilon v_4$	v_3	v_4

Applying the adjoint action of v_1 to

$$v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4\,, \quad (11)$$

we have

$$\begin{aligned} Ad_{\exp(\epsilon_1 v_1)}v &= a_1v_1 + a_2v_2 + a_3(v_3 - \epsilon_1 v_1) + a_4v_4 \\ &= (a_1,a_2,a_3,a_4)\cdot A_1\cdot(v_1,v_2,v_3,v_4)^{\mathrm{T}}\,, \end{aligned} \quad (12)$$

with

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\epsilon_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (13)$$

In a similar way, we can get A_2, A_3 and A_4,

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{\epsilon_2} \end{pmatrix}, \quad A_3 = \begin{pmatrix} e^{\epsilon_3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\epsilon_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (14)$$

So the general adjoint transformation matrix A is given by

$$A = A_1A_2A_3A_4 = \begin{pmatrix} e^{\epsilon_3} & 0 & 0 & 0 \\ 0 & 1 & 0 & -\epsilon_4 \\ -\epsilon_1\, e^{\epsilon_3} & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{\epsilon_2} \end{pmatrix}. \quad (15)$$

2.2 *Construction of a Two-Dimensional Optimal System*

In this subsection, using the algorithm in Ref. [13], a two-dimensional optimal system of the Lie algebra (9) is constructed. First, we briefly recall the method.

Consider an n-dimensional Lie algebra $\mathcal{G}$, which is spanned by the vector fields $\{v_1, v_2, \ldots, v_n\}$. The corresponding n-parameter Lie group of $\mathcal{G}$ is denoted as G. A family of two-dimensional subalgebras $\{\mathfrak{g}_\alpha\}$ form an optimal system denoted as $\mathcal{O}_2$ if: (i) A two-dimensional subalgebra is equivalent to some $\mathfrak{g}_\alpha$, and (ii) $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are inequivalent for distinct α and β. Each element $\mathfrak{g}_\alpha \in \mathcal{O}_2$ is a collection of two linear combinations of the generators. Let

$$\{w_1, w_2\} = \Big(\sum_{i=1}^n a_iv_i, \sum_{j=1}^n b_jv_j\Big)$$

be a general two-dimensional subalgebra, which remains closed under commutation. Two elements $\{w_1, w_2\}$ and $\{w_1', w_2'\}$ are called equivalent if we can find some transformation $g \in G$ and four constants k_1, k_2, k_3, k_4, such that

$$\begin{aligned} w_1' &= k_1Ad_g(w_1) + k_2Ad_g(w_2)\,, \\ w_2' &= k_3Ad_g(w_1) + k_4Ad_g(w_2)\,. \end{aligned} \quad (16)$$

Since w_1' and w_2' are linearly independent, it requires that $k_1k_4 - k_2k_3 \neq 0$ in Eq. (16). In addition, for any element $\{w_1, w_2\} \in \mathcal{O}_2$, it requires that w_1 and w_2 constitute a two-dimensional subalgebra, i.e. $[w_1, w_2] = \lambda w_1 + \mu w_2$ with λ and μ being constants. Galas refines this selection by demonstrating that w_2 must be an element from the normalizer of w_1. That is to say we can select w_2 for $[w_1, w_2] = \lambda w_1$. Furthermore, for any two-dimensional subalgebra $\{w_1, w_2\}$ with $[w_1, w_2] = \lambda w_1$, we can easily find an equivalent one $\{\tilde{w}_1, \tilde{w}_2\}$ with $[\tilde{w}_1, \tilde{w}_2] = 0$ or $[\tilde{w}_1, \tilde{w}_2] = \tilde{w}_1$. Hence, to find the inequivalent elements in the optimal system $\mathcal{O}_2$, without loss of generality, we require each member $\{w_1, w_2\} \in \mathcal{O}_2$ satisfy $[w_1, w_2] = 0$ or $[w_1, w_2] = w_1$. It has been shown that for any two equivalent subalgebras $\{w_1, w_2\}$ and $\{w_1', w_2'\}$, there is $[w_1', w_2'] = 0$ if and only if $[w_1, w_2] = 0$, $[w_1', w_2'] \neq 0$ if and only if $[w_1, w_2] \neq 0$. For the later case, we have the following remark.

Remark 2 If two subalgebras $\{w_1, w_2\}$ and $\{w_1', w_2'\}$ with $[w_1, w_2] = w_1$ and $[w_1', w_2'] = w_1'$ are equivalent in the form of Eq. (16), there must be $k_2 = 0$ and $k_4 = 1$.

To find all the inequivalent elements in the two-dimensional optimal system $\mathcal{O}_2$, we first require each $\{w_1, w_2\} \in \mathcal{O}_2$ satisfy

$$[w_1, w_2] = \delta w_1 , \quad \text{when} \quad \delta \equiv 0, 1 . \tag{17}$$

Thus, when take

$$w_1 = \sum_{i=1}^{n} a_i v_i , \quad w_2 = \sum_{j=1}^{n} b_j v_j , \tag{18}$$

the restriction (17) produces a set of equations of a_i and b_j for two different cases.

For any two-dimensional subalgebra

$$\Big\{ \sum_{i=1}^{n} a_i v_i, \sum_{j=1}^{n} b_j v_j \Big\} ,$$

a $2n$-dimensional function of $\phi(a_1, \ldots, a_n, b_1, \ldots, b_n)$ is called an invariant if it satisfies $\phi(a_{11} Ad_g(w_1) + a_{12} Ad_g(w_2), a_{21} Ad_g(w_1) + a_{22} Ad_g(w_2)) = \phi(w_1, w_2)$ for all $g \in G$ with a_{11}, a_{12}, a_{21}, a_{22} being arbitrary constants.

Taking a general subgroup $g = \exp(\epsilon v)$, ($v = \sum_{k=1}^{n} c_k v_k$) to act on w_1, we have

$$\begin{aligned} Ad_g(w_1) &= Ad_{\exp(\epsilon v)}(w_1) \\ &= w_1 - \epsilon[v, w_1] + \frac{1}{2!}\epsilon^2 [v, [v, w_1]] - \cdots \\ &= (a_1 - \epsilon\Theta_1^a) v_1 + (a_2 - \epsilon\Theta_2^a) v_2 + \cdots \\ &\quad + (a_n - \epsilon\Theta_n^a) v_n + O(\epsilon^2) , \end{aligned} \tag{19}$$

in which $\Theta_i^a \equiv \Theta_i^a(a_1, \ldots, a_n, c_1, \ldots, c_n)$ can be easily get with the commutator table. In the same way, we have

$$\begin{aligned} Ad_g(w_2) &= (b_1 - \epsilon\Theta_1^b) v_1 + (b_2 - \epsilon\Theta_2^b) v_2 + \cdots \\ &\quad + (b_n - \epsilon\Theta_n^b) v_n + O(\epsilon^2) . \end{aligned} \tag{20}$$

More intuitively, the following notations are adopted

$$\begin{aligned} & w_1 \doteq (a_1, a_2, \ldots, a_n) , \quad w_2 \doteq (b_1, b_2, \ldots, b_n) , \\ & Ad_g(w_1) \doteq (a_1 - \epsilon\Theta_1^a, a_2 - \epsilon\Theta_2^a, \ldots, a_n - \epsilon\Theta_n^a) + O(\epsilon^2) , \\ & Ad_g(w_2) \doteq (b_1 - \epsilon\Theta_1^b, b_2 - \epsilon\Theta_2^b, \ldots, b_n - \epsilon\Theta_n^b) + O(\epsilon^2) . \end{aligned} \tag{21}$$

For a two-dimensional subalgebra $\{w_1, w_2\}$, according to the definition of an invariant, we have

$$\begin{aligned} & \phi(a_{11} Ad_g(w_1) + a_{12} Ad_g(w_2), a_{21} Ad_g(w_1) \\ & \quad + a_{22} Ad_g(w_2)) = \phi(w_1, w_2) . \end{aligned} \tag{22}$$

In addition, to guarantee $a_{11} Ad_g(w_1) + a_{12} Ad_g(w_2) = w_1$ and $a_{21} Ad_g(w_1) + a_{22} Ad_g(w_2) = w_2$ after the substitution of $\epsilon = 0$, it requires that

$$\begin{aligned} & a_{11} \equiv 1 + \epsilon a_{11} , \quad a_{12} \equiv \epsilon a_{12} , \\ & a_{21} \equiv \epsilon a_{21} , \quad a_{22} \equiv 1 + \epsilon a_{22} . \end{aligned} \tag{23}$$

Thus Eq. (22) becomes

$$\begin{aligned} & \phi(w_1, w_2) = \phi((1 + \epsilon a_{11}) Ad_g(w_1) \\ & \quad + a_{12} Ad_g(w_2), a_{21} Ad_g(w_1) + (1 + \epsilon a_{22}) Ad_g(w_2)) . \end{aligned} \tag{24}$$

The following two distinct cases need to be considered to determine the invariants ϕ.

(i) When $[w_1, w_2] = 0$, substituting (21) into Eq. (24), then taking the derivative of Eq. (24) with respect to ϵ and setting $\epsilon = 0$, extracting all the coefficients of c_i, a_{11}, a_{12}, a_{21}, a_{22}, a set of linear differential equations about ϕ are achieved. Solving these equations, all the invariants ϕ on $[w_1, w_2] = 0$ can be obtained.

(ii) When $[w_1, w_2] = w_1$, firstly taking $a_{12} = 0$ and $a_{22} = 0$ in Eq. (24), then making the same procedure just as case (i).

For the Lie algebra (9), we take

$$w_1 = \sum_{i=1}^{4} a_i v_i , \quad w_2 = \sum_{j=1}^{4} b_j v_j . \tag{25}$$

Let $v = \sum_{k=1}^{4} c_k v_k$ be a general element from $\mathcal{G}$, in conjunction with Table 1, we have

$$\begin{aligned} Ad_g(w_1) &= Ad_{\exp(\epsilon v)}(w_1) \\ &= (a_1 v_1 + \cdots) + a_4 v_4) - \epsilon[c_1 v_1 + \cdots \\ &\quad + c_4 v_4, a_1 v_1 + \cdots + a_4 v_4] + O(\epsilon^2) \\ &= (a_1 - \epsilon\Theta_1^a) v_1 + \cdots + (a_4 - \epsilon\Theta_4^a) v_4 + O(\epsilon^2) , \end{aligned} \tag{26}$$

where

$$\begin{aligned} & \Theta_1^a = a_3 c_1 - a_1 c_3 , \quad \Theta_2^a = 0 , \\ & \Theta_3^a = 0 , \quad \Theta_4^a = a_2 c_4 - a_4 c_2 . \end{aligned} \tag{27}$$

Similarly, applying $v = \sum_{k=1}^{4} c_k v_k$ to w_2, we get

$$Ad_g(w_2) = (b_1 - \epsilon\Theta_1^b) v_1 + \cdots + (b_4 - \epsilon\Theta_4^b) v_4 + O(\epsilon^2) , \tag{28}$$

with

$$\begin{aligned} & \Theta_1^b = b_3 c_1 - b_1 c_3 , \quad \Theta_2^b = 0 , \quad \Theta_3^b = 0 , \\ & \Theta_4^b = b_2 c_4 - b_4 c_2 . \end{aligned} \tag{29}$$

According to the algorithm, the following two cases need to be considered.

(i) When $[w_1, w_2] = 0$, taking the derivative of Eq. (24) with respect to ϵ and then setting $\epsilon = 0$, extracting all the coefficients of c_i ($i = 1, 2, 3, 4$), a_{11}, a_{12}, a_{21}, a_{22}, eight differential equations of $\phi \equiv \phi(a_1, \ldots, a_4, b_1, \ldots, b_4)$ can be got,

$$\begin{aligned} & a_1 \frac{\partial\phi}{\partial a_1} + b_1 \frac{\partial\phi}{\partial b_1} = 0 , \quad a_2 \frac{\partial\phi}{\partial a_4} + b_2 \frac{\partial\phi}{\partial b_4} = 0 , \\ & a_3 \frac{\partial\phi}{\partial a_1} + b_3 \frac{\partial\phi}{\partial b_1} = 0 , \quad a_4 \frac{\partial\phi}{\partial a_4} + b_4 \frac{\partial\phi}{\partial b_4} = 0 , \end{aligned} \tag{30}$$

$$\begin{aligned} & a_1 \frac{\partial\phi}{\partial a_1} + a_2 \frac{\partial\phi}{\partial a_2} + a_3 \frac{\partial\phi}{\partial a_3} + a_4 \frac{\partial\phi}{\partial a_4} = 0 , \\ & a_1 \frac{\partial\phi}{\partial b_1} + a_2 \frac{\partial\phi}{\partial b_2} + a_3 \frac{\partial\phi}{\partial b_3} + a_4 \frac{\partial\phi}{\partial b_4} = 0 , \\ & b_1 \frac{\partial\phi}{\partial a_1} + b_2 \frac{\partial\phi}{\partial a_2} + b_3 \frac{\partial\phi}{\partial a_3} + b_4 \frac{\partial\phi}{\partial a_4} = 0 , \\ & b_1 \frac{\partial\phi}{\partial b_1} + b_2 \frac{\partial\phi}{\partial b_2} + b_3 \frac{\partial\phi}{\partial b_3} + b_4 \frac{\partial\phi}{\partial b_4} = 0 . \end{aligned} \tag{31}$$

(ii) When $[w_1, w_2] = w_1$, plugging $a_{12} = a_{22} = 0$ into Eq. (24) and making the same process in case (i), six equations about ϕ are obtained, which are just Eqs. (30).

Substituting Eq. (25) into $[w_1, w_2] = \delta w_1$, the following restrictive equations are obtained

$$a_1 b_3 - a_3 b_1 = \delta a_1\,, \quad 0 = \delta a_2\,,$$
$$0 = \delta a_3\,, \quad a_4 b_2 - a_2 b_4 = \delta a_4\,. \tag{32}$$

For two distinct classes $\delta = 0$ and $\delta = 1$, in terms of every restricted condition given by Eqs. (32), compute their respective invariants and select the corresponding eligible representative elements $\{w'_1, w'_2\}$. For ease of calculations, we rewrite Eq. (16) as

$$\begin{cases} Ad_g(w_1) = k'_1 w'_1 + k'_2 w'_2\,, \\ Ad_g(w_2) = k'_3 w'_1 + k'_4 w'_2, \end{cases} \quad (k'_1 k'_4 \neq k'_2 k'_3)\,. \tag{33}$$

More intuitively, Eqs. (33) are usually expressed as

$$\begin{cases} (a_1, a_2, a_3, a_4)A = k'_1(a'_1, a'_2, a'_3, a'_4) + k'_2(b'_1, b'_2, b'_3, b'_4)\,, \\ (b_1, b_2, b_3, b_4)A = k'_3(a'_1, a'_2, a'_3, a'_4) + k'_4(b'_1, b'_2, b'_3, b'_4)\,, \end{cases} \quad (k'_1 k'_4 \neq k'_2 k'_3)\,, \tag{34}$$

where the general adjoint transformation matrix A is given in Eq. (15). If Eqs. (34) have solution with repect to $\epsilon_i, k'_i, (i = 1, 2, 3, 4)$, it implies that the selected representative element $\{w'_1, w'_2\}$ is correct; if Eqs. (34) have no solution, another representative element $\{w''_1, w''_2\}$ need to be selected. Repeat the process until all the cases are finished.

(i) The case of $\delta = 0$ in the restrictive equations (32).

Substituting $\delta = 0$ into Eqs. (32), we have

$$a_1 b_3 - a_3 b_1 = 0\,, \quad a_4 b_2 - a_2 b_4 = 0\,. \tag{35}$$

For this case, two different situations need to be considered.

Case 1 Not all a_3 and b_3 are zeros. Without loss of generality, we take $a_3 \neq 0$, then $b_1 = a_1(b_3/a_3)$. In this case, there exist three subclasses.

Case 1.1 When $a_2 \neq 0$, then we have $b_4 = a_4(b_2/a_2)$. Substituting $b_1 = a_1(b_3/a_3)$ and $b_4 = a_4(b_2/a_2)$ into Eqs. (30) and Eqs. (31), we find that $\phi =$ constant. The representative element $\{v_2, v_3\}$ can be selected, since Eqs. (34) have the solution

$$k'_1 = a_2\,, \quad k'_2 = a_3\,, \quad k'_3 = b_2\,,$$
$$k'_4 = b_3\,, \quad \epsilon_1 = \frac{a_1}{a_3}\,, \quad \epsilon_4 = \frac{a_4}{a_2}\,\mathrm{e}^{\epsilon_2}\,.$$

Case 1.2 When $a_2 = 0$, $b_2 \neq 0$, then it must have $a_4 = 0$. In a similar way, we find that $\phi =$ constant. We select the representative element $\{v_3, v_2\}$, then Eqs. (34) have the solution

$$k'_1 = a_3\,, \quad k'_2 = 0\,, \quad k'_3 = b_3$$
$$k'_4 = b_2, \quad \epsilon_1 = \frac{a_1}{a_3}\,, \quad \epsilon_4 = \frac{b_4}{b_2}\,\mathrm{e}^{\epsilon_2}\,.$$

It is obvious that Cases 1.1 and 1.2 are equivalent.

Case 1.3 When $a_2 = b_2 = 0$. In this case, $\{v_4, v_3\}$ can be selected as the representative element since Eqs. (34) hold for

$$k'_1 = a_4\,\mathrm{e}^{\epsilon_2}\,, \quad k'_2 = a_3\,, \quad k'_3 = b_4\,\mathrm{e}^{\epsilon_2}\,,$$
$$k'_4 = b_3\,, \quad \epsilon_1 = \frac{a_1}{a_3}\,.$$

Case 2 When $a_3 = b_3 = 0$. This case can be further divided into two subclasses.

Case 2.1 Not all a_2 and b_2 are zeros. Without loss of generality, we take $a_2 \neq 0$, then $b_4 = a_4(b_2/a_2)$. We choose a representative element $\{v_1, v_2\}$, then Eqs. (34) have the solution

$$k'_1 = a_1\,\mathrm{e}^{\epsilon_3}\,, \quad k'_2 = a_2\,, \quad k'_3 = b_1\,\mathrm{e}^{\epsilon_3}\,,$$
$$k'_4 = b_2\,, \quad \epsilon_4 = \frac{a_4}{a_2}\,\mathrm{e}^{\epsilon_2}\,.$$

Case 2.2 When $a_2 = b_2 = 0$, $\{v_1, v_4\}$ can be selected as the representative element since Eqs. (34) hold for

$$k'_1 = a_1\,\mathrm{e}^{\epsilon_3}\,, \quad k'_2 = a_4\,\mathrm{e}^{\epsilon_2}\,, \quad k'_3 = b_1\,\mathrm{e}^{\epsilon_3}\,, \quad k'_4 = b_4\,\mathrm{e}^{\epsilon_2}\,.$$

(ii) The case of $\delta = 1$ in the restrictive equations (32).

Substituting $\delta = 1$ into Eqs. (32), it must have $a_2 = a_3 = 0$. Then Eqs. (32) become to

$$a_1 b_3 = a_1\,, \quad a_4 b_2 = a_4\,. \tag{36}$$

Case 3 When $a_1 = 0$, it must have $a_4 \neq 0$, thus we get $b_2 = 1$. In this case, by solving Eqs. (30), we get an invariant $\phi = \Delta_1 = b_3$. This case can be further divided into two subclasses according to the values of Δ_1.

Case 3.1 If $\Delta_1 = \alpha \neq 0$, we can select the representative element $\{v_4, v_2 + \alpha v_3\}$, then Eqs. (34) have the solution

$$k'_1 = a_4\,\mathrm{e}^{\epsilon_2}\,, \quad k'_2 = 0\,, \quad k'_3 = b_4\,\mathrm{e}^{\epsilon_2} - \epsilon_4\,,$$
$$k'_4 = 1\,, \quad \epsilon_1 = \frac{b_1}{\alpha}\,.$$

Case 3.2 If $\Delta_1 = \alpha = 0$, there exist three circumstances in terms of the sign of b_1.

(a) When $b_1 > 0$, $\{v_4, v_1 + v_2\}$ can be chosen as the representative element since Eqs. (34) hold for

$$k'_1 = a_4\,\mathrm{e}^{\epsilon_2}\,, \quad k'_2 = 0\,, \quad k'_3 = b_4\,\mathrm{e}^{\epsilon_2} - \epsilon_4\,,$$
$$k'_4 = 1\,, \quad \epsilon_3 = -\ln(b_1)\,.$$

(b) When $b_1 < 0$, we select the representative element $\{v_4, -v_1 + v_2\}$ and Eqs. (34) have the solution

$$k'_1 = a_4\,\mathrm{e}^{\epsilon_2}\,, \quad k'_2 = 0\,, \quad k'_3 = b_4\,\mathrm{e}^{\epsilon_2} - \epsilon_4\,,$$
$$k'_4 = 1, \quad \epsilon_3 = -\ln(-b_1)\,.$$

(c) When $b_1 = 0$, we select the representative element $\{v_4, v_2\}$ and Eqs. (34) hold for

$$k'_1 = a_4\,\mathrm{e}^{\epsilon_2}\,, \quad k'_2 = 0\,, \quad k'_3 = b_4\,\mathrm{e}^{\epsilon_2} - \epsilon_4\,, \quad k'_4 = 1.$$

Case 4 When $a_1 \neq 0$, $a_4 = 0$, then we have $b_3 = 1$. In this case, by solving Eqs. (30), we get an invariant $\phi = \Delta_2 = b_2$. This case can also be further divided into two subclasses according to the values of Δ_2.

Case 4.1 If $\Delta_2 = \alpha \neq 0$, we select the representative element $\{v_1, \alpha v_2 + v_3\}$, then Eqs. (34) have the solution

$$k_1' = a_1 e^{\epsilon_3}\,, \quad k_2' = 0\,, \quad k_3' = (b_1 - \epsilon_1)e^{\epsilon_3}\,,$$
$$k_4' = 1, \quad \epsilon_4 = \frac{b_4}{\alpha} e^{\epsilon_2}\,.$$

Case 4.2 If $\Delta_2 = \alpha = 0$, there exist three circumstances in terms of the sign of b_4.

(a) When $b_4 > 0$, $\{v_1, v_3 + v_4\}$ can be chosen as the representative element since Eqs. (34) hold for

$$k_1' = a_1 e^{\epsilon_3}\,, \quad k_2' = 0\,, \quad k_3' = (b_1 - \epsilon_1)e^{\epsilon_3}\,,$$
$$k_4' = 1, \quad \epsilon_2 = -\ln(b_4)\,.$$

(b) When $b_4 < 0$, we select the representative element $\{v_1, v_3 - v_4\}$ and Eqs. (34) have the solution

$$k_1' = a_1 e^{\epsilon_3}\,, \quad k_2' = 0\,, \quad k_3' = (b_1 - \epsilon_1)e^{\epsilon_3}\,,$$
$$k_4' = 1\,, \quad \epsilon_2 = -\ln(-b_4)\,.$$

(c) When $b_4 = 0$, we select the representative element $\{v_1, v_3\}$ and Eqs. (34) have the solution

$$k_1' = a_1 e^{\epsilon_3}\,, \quad k_2' = 0\,, \quad k_3' = (b_1 - \epsilon_1)e^{\epsilon_3}\,, \quad k_4' = 1\,.$$

Case 5 When $a_1 \neq 0$, $a_4 \neq 0$, then we have $b_2 = b_3 = 1$. Solving Eqs. (30), we find that $\phi =$ constant. In this case, there exist two circumstances in terms of the sign of $a_1 a_4$.

(a) When $a_1 a_4 > 0$, we select the representative element $\{v_1 + v_4, v_2 + v_3\}$ and Eqs. (34) have the solution

$$k_1' = a_1\, e^{\epsilon_3}\,, \quad k_2' = 0\,, \quad k_3' = \frac{a_1 b_4}{a_4} e^{\epsilon_3} - \epsilon_4\,, \quad k_4' = 1\,,$$
$$\epsilon_2 = \ln\Bigl(\frac{a_1}{a_4}\Bigr) + \epsilon_3\,, \quad \epsilon_4 = \frac{a_1 b_4 - a_4 b_1 + a_4 \epsilon_1}{a_4} e^{\epsilon_3}\,.$$

(b) When $a_1 a_4 < 0$, $\{v_1 - v_4, v_2 + v_3\}$ can be chosen as the representative element since Eqs. (34) hold for

$$k_1' = a_1\, e^{\epsilon_3}\,, \quad k_2' = 0\,, \quad k_3' = \frac{a_1 b_4}{a_4} e^{\epsilon_3} + \epsilon_4\,, \quad k_4' = 1\,,$$
$$\epsilon_2 = \ln\Bigl(-\frac{a_1}{a_4}\Bigr) + \epsilon_3\,, \quad \epsilon_4 = -\frac{a_1 b_4 - a_4 b_1 + a_4 \epsilon_1}{a_4} e^{\epsilon_3}\,.$$

In summary, a two-dimensional optimal system $\mathcal{O}_2$ of the Lie algebra (9) is obtained

$$\begin{aligned}
&\mathfrak{g}_1 = \{v_2, v_3\}\,, \quad \mathfrak{g}_2 = \{v_3, v_4\}\,, \quad \mathfrak{g}_3 = \{v_1, v_2\}\,,\\
&\mathfrak{g}_4 = \{v_1, v_4\}\,, \mathfrak{g}_5 = \{v_2 + \alpha v_3, v_4\}\,, \quad \mathfrak{g}_6 = \{v_1 + v_2, v_4\}\,,\\
&\mathfrak{g}_7 = \{-v_1 + v_2, v_4\}\,, \quad \mathfrak{g}_8 = \{v_2, v_4\}\,,\\
&\mathfrak{g}_9 = \{v_1, \alpha v_2 + v_3\}\,, \quad \mathfrak{g}_{10} = \{v_1, v_3 + v_4\}\,,\\
&\mathfrak{g}_{11} = \{v_1, v_3 - v_4\}\,, \quad \mathfrak{g}_{12} = \{v_1, v_3\}\,,\\
&\mathfrak{g}_{13} = \{v_1 + v_4, v_2 + v_3\}\,,\\
&\mathfrak{g}_{14} = \{v_1 - v_4, v_2 + v_3\}\,, \quad (\alpha \neq 0, \alpha \in \mathbb{R})\,.
\end{aligned} \tag{37}$$

The value in parentheses does not denote disallowed value, it denotes the value of the parameter that needs not to be considered, because it is considered elsewhere. This information is important in the calculation of the similarity variables.

3 Symmetry Reductions and Exact Solutions

By virtue of the two-dimensional optimal system $\mathcal{O}_2$ (37), the boundary-layer equations (2) and (3) can be directly reduced to different classes of ODEs. This is achieved by solving the invariant surface condition equations to find all the similarity invariants, which are then used as new variables.

For $\mathfrak{g}_1 = \{v_2, v_3\}$, the invariant surface condition equations are given by

$$\begin{aligned}
&x u_x - u = 0\,, \quad t u_t + \frac{1}{2} y u_y + u = 0\,,\\
&x v_x = 0\,, \quad t v_t + \frac{1}{2} y v_y + \frac{1}{2} v = 0\,,\\
&x w_x - w = 0\,, \quad t w_t + \frac{1}{2} y w_y + w = 0\,.
\end{aligned} \tag{38}$$

Solving Eq. (38), we have

$$u = \frac{x}{t} U(z)\,, \quad v = \frac{1}{\sqrt{t}} V(z)\,, \quad w = \frac{x}{t} W(z)\,, \tag{39}$$

where $z = y/\sqrt{t}$ is a similarity variable. Substituting Eq. (39) into Eqs. (2), a system of ODEs is obtained

$$\begin{aligned}
&U + V' = 0\,,\\
&U'' + \Bigl(\frac{z}{2} - V\Bigr) U' + (W + U - 1)(W - U) = 0\,,\\
&W' = 0\,.
\end{aligned} \tag{40}$$

The boundary conditions (3) then require

$$U(0) = V(0) = 0\,, \quad U(z) = W(z)\,, \quad \text{as} \quad z \to \infty\,. \tag{41}$$

From the third equation in Eq. (40), we conclude that $W(z) = E$, E is a constant. Using the transformation $V(z) = -E f(z)$, from Eqs. (39)–(41), we can get

$$u = \frac{Ex}{t} f'(z)\,, \; v = -\frac{E}{\sqrt{t}} f(z)\,, \; w = \frac{Ex}{t}\,, \; z = \frac{y}{\sqrt{t}}\,, \tag{42}$$

where $f(z)$ satisfies

$$\begin{aligned}
&f''' + \Bigl(\frac{z}{2} + E f\Bigr) f'' + (1 - E f') f' + E - 1 = 0\,,\\
&f(0) = f'(0) = 0\,, \qquad f'(\infty) = 1\,.
\end{aligned} \tag{43}$$

This solution has been obtained by Ma and Hui, they refer this solution as the unsteady separated staparated-point flow solution (USSP) and study it in detail.

In a similar way, for $\mathfrak{g}_2 = \{v_3, v_4\}$, an analytic solution can be obtained

$$\begin{aligned}
&u = E\frac{z}{t} \exp\Bigl(-\frac{z^2}{4}\Bigr)\Bigl(C + \int_0^{z/2} \exp(\tau^2)\,\mathrm{d}\tau\Bigr)\,,\\
&v = 0\,, \quad w = \frac{E}{t}\,, \quad z = \frac{y}{\sqrt{t}}\,,
\end{aligned} \tag{44}$$

where C is an arbitrary constant. It is easy to verify that this solution also satisfies the Navier–Stokes equations (1) with the pressure given by

$$p = E\frac{x}{t^2} + p_0(y)\,. \tag{45}$$

For $\mathfrak{g}_3 = \{v_1, v_2\}$, it is easy to get

$$u = Exf'(y)\,, \quad v = -Ef(y)\,, \quad w = Ex\,, \tag{46}$$

where $f(y)$ satisfies

$$f''' + E(ff'' - f'^2 + 1) = 0\,,$$
$$f(0) = f'(0) = 0\,, \quad f'(\infty) = 1\,, \tag{47}$$

which is the Hiemenz stagnation-point flow solution.

By virtue of $\mathfrak{g}_5 = \{v_2 + \alpha v_3, v_4\}$ $(\alpha \neq 0)$, we can get that

$$u = Et^q f(z)\,, \quad v = 0\,, \quad w = Et^q\,, \quad z = \frac{y}{\sqrt{t}}\,, \tag{48}$$

with $f(z)$ satisfies

$$f'' + \frac{z}{2}f' + q(1 - f) = 0\,, \quad f(0) = 0\,, \quad f(\infty) = 1\,, \tag{49}$$

where $q = 1/\alpha - 1$. The solution given by Eqs. (48) and (49) also satisfies the Navier–Stokes equations (1) with the pressure

$$p = p_0(y) - Exqt^{q-1}\,. \tag{50}$$

An analytic solution of Eq. (49) is given in Ref. [20] when q is a positive integer. Here, we consider three special cases when q is not a positive integer.

When $\alpha = 1$ $(q = 0)$, Eq. (49) admits an error function solution

$$f(z) = \mathrm{erf}\Big(\frac{z}{2}\Big)\,, \tag{51}$$

where the error function $\mathrm{erf}(x)$ is defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\,\mathrm{d}t\,. \tag{52}$$

This case results in the Rayleigh solution.

When $\alpha = 2/3$ $(q = 1/2)$, Eq. (49) has a solution of the form

$$f(z) = \frac{z\sqrt{\pi}}{2}\Big(1 - \mathrm{erf}\Big(\frac{z}{2}\Big)\Big) - \exp\Big(-\frac{z^2}{4}\Big) + 1\,. \tag{53}$$

In this case, the stream function Ψ of the flow Eq. (48) is given by

$$\Psi = E\Big[\frac{\sqrt{\pi}}{4}y^2 - \frac{\sqrt{\pi}}{4}\mathrm{erf}\Big(\frac{1}{2}\frac{y}{\sqrt{t}}\Big)(2t + y^2)$$
$$+ y\sqrt{t}\Big(1 - \frac{1}{2}\exp\Big(-\frac{1}{4}\frac{y^2}{t}\Big)\Big)\Big]\,. \tag{54}$$

When $\alpha = 2$ $(q = -1/2)$, another solution of Eq. (49) is obtained

$$f(z) = 1 - \exp\Big(-\frac{z^2}{4}\Big)\,. \tag{55}$$

For this case, the stream function Ψ is given by

$$\Psi = E\Big[\frac{y}{\sqrt{t}} - \sqrt{\pi}\,\mathrm{erf}\Big(\frac{1}{2}\frac{y}{\sqrt{t}}\Big)\Big]\,. \tag{56}$$

The solutions (53) and (55) are not reported in Ref. [20]. More analytic solutions to Eq. (49) for other values of α can also be obtained, but for brevity, we do not present here.

From $\mathfrak{g}_6 = \{v_1 + v_2, v_4\}$, the following solution can be achieved

$$u = E\mathrm{e}^t(1 - \mathrm{e}^{-y})\,, \quad v = 0\,, \quad w = E\mathrm{e}^t\,. \tag{57}$$

The solution (57) also satisfies the Navier–Stokes equations (1) with the pressure

$$p = p_0(y) - Ex\mathrm{e}^t\,. \tag{58}$$

For $\mathfrak{g}_{13} = \{v_1 + v_4, v_2 + v_3\}$, we get

$$u = Ef(z)\,, \quad v = \frac{E}{\sqrt{t - x}}g(z)\,,$$
$$w = E\,, \quad z = \frac{y}{\sqrt{t - x}}\,, \tag{59}$$

where $f(z)$ and $g(z)$ satisfy

$$g' + \frac{z}{2}f' = 0\,,$$
$$f'' + \frac{z}{2}f'(1 - Ef) - Egf' = 0\,,$$
$$f(0) = g(0) = 0\,, \quad f(\infty) = 1\,. \tag{60}$$

In addition, by the transformation $f(z) = (1/E)(1 - h'(z))$, Eq. (60) can be converted to

$$h''' + \frac{1}{2}(h - C)h'' = 0\,, \quad h(0) = C\,,$$
$$h'(0) = 1\,, \quad h'(\infty) = 1 - E\,. \tag{61}$$

With $g(z)$ is given by

$$g(z) = \frac{1}{2E}(zh'(z) - h(z) + C)\,. \tag{62}$$

From Eq. (62), it is obvious that $g(0) = 0$.

For $\mathfrak{g}_{14} = \{v_1 - v_4, v_2 + v_3\}$, we have

$$u = Ef(z)\,, \quad v = \frac{E}{\sqrt{t + x}}g(z)\,,$$
$$w = E\,, \quad z = \frac{y}{\sqrt{t + x}}\,, \tag{63}$$

where $f(z)$ and $g(z)$ satisfy

$$g' - \frac{z}{2}f' = 0\,,$$
$$f'' + \frac{z}{2}f'(1 + Ef) - Egf' = 0\,,$$
$$f(0) = g(0) = 0\,, \quad f(\infty) = 1\,. \tag{64}$$

Similarly, by the transformation $f(z) = (1/E)(h'(z) - 1)$, Eqs. (64) become to

$$h''' + \frac{1}{2}(h - C)h'' = 0\,, \quad h(0) = C\,,$$
$$h'(0) = 1\,, \quad h'(\infty) = 1 + E\,. \tag{65}$$

In this case, $g(z)$ also meets (62). These two reductions from $\mathfrak{g}_{13}$ and $\mathfrak{g}_{14}$ are not obtained in Ref. [20], as far as we know, they may be previously unknown.

More reductions and exact solutions of the boundary-layer equations can be found by virtue of the other elements in the optimal system (37). For instants, using

$\mathfrak{g}_9, \mathfrak{g}_{10}, \mathfrak{g}_{11}$ and $\mathfrak{g}_{12}$, the steady boundary-layer flow solutions may be obtained. For volumes, we do not present here as all the existing group invariant steady solutions have been listed in Ref. [20], and no new group invariant solutions can be found. While only trivial solution can be found from $\mathfrak{g}_4, \mathfrak{g}_7$ and $\mathfrak{g}_8$.

4 Conclusion

In conclusion, the boundary-layer equations which are important models in fluid mechanics are studied through the classical Lie symmetry method. Its symmetry group is narrowed down to a subgroup under which the boundary conditions are also invariant. To find intrinsically different similarity reductions and inequivalent group invariant solutions, a two-dimensional optimal system is constructed. Since all the representative elements in the optimal system are attached to different values of the invariants, it can ensure the optimality of the optimal system obtained, need no further proofs. We notice that in almost all of the existing literatures, a one-dimensional optimal system is required for the construction of a two-dimensional optimal system, which usually takes too much work. In this paper, the construction starts from the Lie algebra directly and only depends on fragments of the theory of Lie algebras, without a prior one-dimensional optimal system. Then with the aid of the optimal system, some symmetry reductions and exact solutions of the boundary-layer equations are obtained. It has been shown that not only do we recover many of the known results but also find some new solutions, which may be previously unknown. In Ref. [20], the authors investigated the boundary-layer equations using a two-step reduction procedure, and some of the reductions obtained are overlapped. In our paper, due to the two-dimensional optimal system, the original equations are reduced to a system of ODEs via only one step. What is more, the reduced systems are intrinsically different and the solutions obtained are inequivalent, and some repetitive works are also avoided.

As we know, the nonclassical symmetry (also known as conditional symmetry) method generalizes and includes the classical method in studying reductions and solutions of PDEs.[5] So the nonclassical symmetry analysis of the boundary-layer equations may generate more new solutions, which is interesting and deserves our further study.

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