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Nonlocal symmetry and similarity reductions for the Drinfeld–Sokolov–Satsuma–Hirota system

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ABSTRACT

The nonlocal symmetry of the Drinfeld–Sokolov–Satsuma–Hirota system is obtained from the known Lax pair, and infinitely many nonlocal symmetries are given by introducing the internal parameters. Then the nonlocal symmetry is localized to a prolonged system by introducing suitable auxiliary dependent variables. By applying the classical Lie symmetry method to this prolonged system, two main results are obtained: a new type of finite symmetry transformation is derived, which can generate new solutions from old ones; some exact interaction solutions among solitons and other complicated waves including periodic cnoidal wave and Painlevé waves are derived through similarity reductions.

1. Introduction

The Drinfeld–Sokolov–Satsuma–Hirota (DSSH) system of coupled nonlinear evolution equations

$$
\begin{aligned}
u_t &= \frac{1}{2}u_{xxx} - 3uu_x + 3v_x, \\
v_t &= -v_{xxx} + 3uv_x,
\end{aligned}
\tag{1}
$$

which was proposed, independently, by Drinfeld and Sokolov [1], and by Satsuma and Hirota [2]. In [1], the system (1) was developed as one example of nonlinear equations possessing Lax pairs of a special form. In [2], this system was found as a special case of the four-reduction of the KP hierarchy, and its one-soliton solution was given. In [3], a recursion operator and a bi-Hamiltonian structure for system (1) was obtained, which provided the system with an infinite algebra of generalized symmetries and an infinite set of conservations. In [4], an explicit Bäcklund transformation, which is in fact a superposition of two simple Bäcklund transformations shown in [5], was constructed to derive special solutions of this system by

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the truncated singular expansions method. In [6], the sine–cosine method and the tanh method were used to obtain exact traveling wave solutions. In [7], a class of sixth-order nonlinear wave equations was discussed, which contains the DSSH system (1) as a special case by the Painlevé analysis. In [8], the Cole–Hopf transformation, the tanh–coth method, and the exp-function method were used to obtain multiple singular soliton solutions and singular periodic solutions. In [9], this system was shown to be one of the three nontrivial reductions from a generalized Hirota–Satsuma coupled KdV equation. In [10], a Darboux transformation was constructed with the aid of gauge transformation between the Lax pairs and some soliton solutions, periodic solutions, rational solutions were obtained. In [11], the truncated Painlevé expansion was developed to construct Bäcklund transformations, nonlocal symmetries and the soliton–cnoidal wave solutions were explicitly obtained by the consistent Riccati expansion.

In this paper, we focus on the nonlocal symmetries [12–14] and similarity reductions of the DSSH system (1). Compared with [11], infinitely many nonlocal symmetries, similarity reductions, and group invariant solutions are obtained. The paper is organized as follows. In Section 2, the nonlocal symmetry of the DSSH system is obtained from the Lax pair. Then the nonlocal symmetry is localized to Lie point symmetry by prolonging the original system to a large system. In Section 3, the finite symmetry transformations and similar reductions of the prolonged system are presented, and several new exact solutions of the original system are derived. The last section contains a short summary and discussion.

2. Nonlocal symmetry and its localization

The Lax pair of the DSSH system (1) reads

$$\psi_{xx}=\begin{pmatrix} u+\lambda & v \\ 1 & u-\lambda \end{pmatrix}\psi,\quad \psi_t=\begin{pmatrix} \frac{1}{2}u_x & -v_x \\ 0 & \frac{1}{2}u_x \end{pmatrix}\psi+\begin{pmatrix} -u+2\lambda & 2v \\ 2 & -u-2\lambda \end{pmatrix}\psi_x,$$

we can rewrite the Lax pair in the following form

$$\psi_{1xx}=(\lambda+u)\psi_1+v\psi_2,\quad \psi_{1t}=\frac{1}{2}u_x\psi_1-v_x\psi_2+(-u+2\lambda)\psi_{1x}+2v\psi_{2x}, \tag{2}$$

$$\psi_{2xx}=\psi_1-(\lambda-u)\psi_2,\quad \psi_{2t}=\frac{1}{2}u_x\psi_2+2\psi_{1x}-(u+2\lambda)\psi_{2x}, \tag{3}$$

where $\{u,v\}$ is a solution of system (1), $\{\psi_1,\psi_2\}$ is the spectral function and λ is a spectral parameter.

Proposition 1. *If ψ_1,ψ_2 satisfy Lax pair (2)–(3) with $\lambda=0$, then*

$$\sigma=(\sigma^u,\sigma^v)\equiv(-2\psi_2\psi_{2x},\psi_1\psi_{2x}-\psi_2\psi_{1x}), \tag{4}$$

is a nonlocal symmetry of the DSSH system (1).

Remark 1. It is a fact that if ψ_1,ψ_2 satisfy Lax pair (2)–(3) with the arbitrary spectral parameter λ, σ given by (4) is still a symmetry of the DSSH system (1). This fact can be verified by direct calculation.

From the method used in Refs. [15] and [16], more symmetries were constructed by differentiating a known one with respect to inner parameters. Then one has the following proposition.

Proposition 2. *If a λ-dependent function $\sigma_0(\lambda)$ is a symmetry of the DSSH system (1) with $\lambda\equiv\lambda_1,\lambda_2,\ldots,\lambda_r$, then*

$$\sigma_n\equiv\frac{d^{\{n\}}}{d\lambda^{\{n\}}}\sigma_0(\lambda)\equiv\frac{d^{\{n_1\}}}{d\lambda_1^{\{n_1\}}}\frac{d^{\{n_2\}}}{d\lambda_2^{\{n_2\}}}\cdots\frac{d^{\{n_r\}}}{d\lambda_r^{\{n_r\}}}\sigma_0(\lambda), \tag{5}$$

is also a symmetry of the same DSSH system (1) for $\{n\}\equiv\{n_1,n_2,\ldots,n_r\}$.

By Propositions 1 and 2, one can obtain infinitely many new nonlocal symmetries. For example, if we take $\{\psi_1, \psi_2\}$ and $\{\bar{\psi}_1, \bar{\psi}_2\}$ are two solutions of Lax pair (2)–(3), then

$$\sigma(\lambda_1, \lambda_2) \equiv (\sigma^u(\lambda_1, \lambda_2), \sigma^v(\lambda_1, \lambda_2)), \tag{6}$$

with

$$\begin{aligned}\sigma^u(\lambda_1, \lambda_2) &= -2(\lambda_1\psi_2 + \lambda_2\bar{\psi}_2)(\lambda_1\psi_2 + \lambda_2\bar{\psi}_2)_x,\\ \sigma^v(\lambda_1, \lambda_2) &= (\lambda_1\psi_1 + \lambda_2\bar{\psi}_1)(\lambda_1\psi_2 + \lambda_2\bar{\psi}_2)_x - (\lambda_1\psi_2 + \lambda_2\bar{\psi}_2)(\lambda_1\psi_1 + \lambda_2\bar{\psi}_1)_x,\end{aligned}$$

and $\frac{\partial^{n_1+n_2}}{\partial\lambda_1^{n_1}\partial\lambda_2^{n_2}}\sigma(\lambda_1, \lambda_2)$ are also symmetries of the same DSSH system (1).

By introducing new dependent variables $\phi_1 \equiv \phi_1(x, t)$ and $\phi_2 \equiv \phi_2(x, t)$ with

$$\phi_1 = \psi_{1x}, \quad \phi_2 = \psi_{2x}, \tag{7}$$

the above symmetry (4) is converted into

$$\sigma^u = -2\psi_2\phi_2, \quad \sigma^v = \psi_1\phi_2 - \psi_2\phi_1. \tag{8}$$

In order to compute local symmetries for the variables ψ_1, ψ_2, ϕ_1, and ϕ_2, we have to introduce another potential variable $p \equiv p(x, t)$. The variable p makes the prolonged system closed completely, and satisfies the compatibility conditions:

$$p_x = \psi_2^2, \quad p_t = u\psi_2^2 - 2\phi_2^2 + 4\psi_1\psi_2. \tag{9}$$

Then it can yield

$$\sigma^{\psi_1} = -\frac{1}{2}p\psi_1, \quad \sigma^{\psi_2} = -\frac{1}{2}p\psi_2, \quad \sigma^{\phi_1} = -\frac{1}{2}(\psi_1\psi_2^2 + p\phi_1), \quad \sigma^{\phi_2} = -\frac{1}{2}(\psi_2^3 + p\phi_2), \quad \sigma^p = -\frac{1}{2}p^2, \tag{10}$$

where $\sigma^{\psi_1}, \sigma^{\psi_2}, \sigma^{\phi_1}, \sigma^{\phi_2}$, and σ^p denote the symmetries of $\psi_1, \psi_2, \phi_1, \phi_2$, and p, respectively.

Finally, the prolongation for nonlocal symmetry (4) is successfully localized with the vector form

$$\begin{aligned}V = &-2\psi_2\phi_2\frac{\partial}{\partial u} + (\psi_1\phi_2 - \psi_2\phi_1)\frac{\partial}{\partial v} - \frac{1}{2}p\psi_1\frac{\partial}{\partial\psi_1} - \frac{1}{2}p\psi_2\frac{\partial}{\partial\psi_2}\\ &- \frac{1}{2}(\psi_1\psi_2^2 + p\phi_1)\frac{\partial}{\partial\phi_1} - \frac{1}{2}(\psi_2^3 + p\phi_2)\frac{\partial}{\partial\phi_2} - \frac{1}{2}p^2\frac{\partial}{\partial p}.\end{aligned} \tag{11}$$

Another meaningful point is that the introduced potential variable p just satisfies the Schwartz form of (1)

$$2C_t + 2CC_x - 2CS_x - SS_x + C_{xxx} - S_{xxx} = 0, \tag{12}$$

where $C = \frac{p_t}{p_x}$ and $S = \frac{p_{xxx}}{p_x} - \frac{3p_{xx}^2}{2p_x^2}$ are all invariant under the Möbious transformation with transformation (15)

$$p \to \frac{a + bp}{c + dp} \quad (ad \neq bc). \tag{13}$$

Remark 2. The Schwartz form of a given differential equation is usually derived by utilizing singularity analysis method. The above result is consistent with the Schwartz form [11] reduced by the truncated Painlevé expansion.

3. Explicit solutions from nonlocal symmetry

After making the nonlocal symmetry (4) equivalent to Lie point symmetry (11), one can construct the explicit solutions by Lie group theory in two aspects.

3.1. Finite symmetry transformation

According to Lie point symmetry (11), by solving the following initial value problem:

$$\begin{aligned}
&\frac{d\bar{u}(\epsilon)}{d\epsilon}=-2\bar{\psi}_2\bar{\phi}_2,\quad \frac{d\bar{v}(\epsilon)}{d\epsilon}=\bar{\psi}_1\bar{\phi}_2-\bar{\psi}_2\bar{\phi}_1,\quad \frac{d\bar{\psi}_1(\epsilon)}{d\epsilon}=-\frac{1}{2}\bar{p}\bar{\psi}_1,\quad \frac{d\bar{\psi}_2(\epsilon)}{d\epsilon}=-\frac{1}{2}\bar{p}\bar{\psi}_2,\\
&\frac{d\bar{\phi}_1(\epsilon)}{d\epsilon}=-\frac{1}{2}(\bar{\psi}_1\bar{\psi}_2^2+\bar{p}\bar{\phi}_1),\quad \frac{d\bar{\phi}_2(\epsilon)}{d\epsilon}=-\frac{1}{2}(\bar{\psi}_2^3+\bar{p}\bar{\phi}_2),\quad \frac{d\bar{p}(\epsilon)}{d\epsilon}=-\frac{1}{2}\bar{p}^2,\\
&\bar{u}(0)=u,\quad \bar{v}(0)=v,\quad \bar{\psi}_1(0)=\psi_1,\quad \bar{\psi}_2(0)=\psi_2,\quad \bar{\phi}_1(0)=\phi_1,\quad \bar{\phi}_2(0)=\phi_2,\quad \bar{p}(0)=p,
\end{aligned}\tag{14}$$

where ϵ is the group parameter, we arrive at the following symmetry group theorem:

Theorem 1. *If $\{u, v, \psi_1, \psi_2, \phi_1, \phi_2, p\}$ is a solution of the prolonged system consisting of (1)–(3), (7), and (9) with $\lambda = 0$, then so is $\{\bar{u}, \bar{v}, \bar{\psi}_1, \bar{\psi}_2, \bar{\phi}_1, \bar{\phi}_2, \bar{p}\}$ given by*

$$\begin{aligned}
&\bar{u}=u-\frac{4\epsilon\psi_2\phi_2}{2+\epsilon p}+\frac{2\epsilon^2\psi_2^4}{(2+\epsilon p)^2},\quad \bar{v}=v+\frac{2\epsilon(\psi_1\phi_2-\psi_2\phi_1)}{2+\epsilon p},\quad \bar{\psi}_1=\frac{2\psi_1}{2+\epsilon p},\\
&\bar{\psi}_2=\frac{2\psi_2}{2+\epsilon p},\quad \bar{\phi}_1=\frac{2\phi_1}{2+\epsilon p}-\frac{2\epsilon\psi_1\psi_2^2}{(2+\epsilon p)^2},\quad \bar{\phi}_2=\frac{2\phi_2}{2+\epsilon p}-\frac{2\epsilon\psi_2^3}{(2+\epsilon p)^2},\quad \bar{p}=\frac{2p}{2+\epsilon p}.
\end{aligned}\tag{15}$$

Remark 3. For a given solution $\{u, v\}$ of (1), the above finite symmetry transformation will denote another solution $\{\bar{u}, \bar{v}\}$. It is necessary to point out that the last equation of (15) is nothing but the corresponding Möbious transformation.

3.2. Similarity reductions of the prolonged system

In order to find similarity reductions of (1), we employ the Lie symmetry method to the whole prolonged system. Supposing Eqs. (1)–(3), (7), and (9) are invariant under the infinitesimal transformations

$$\begin{aligned}
\{x,t,u,v,\psi_1,\psi_2,\phi_1,\phi_2,p\}\to\{&x+\epsilon X,t+\epsilon T,u+\epsilon U,v+\epsilon V,\psi_1+\epsilon\Psi_1,\psi_2\\
&+\epsilon\Psi_2,\phi_1+\epsilon\Phi_1,\phi_2+\epsilon\Phi_2,p+\epsilon P\}
\end{aligned}\tag{16}$$

with

$$\begin{aligned}
&\sigma^u=Xu_x+Tu_t-U,\quad \sigma^v=Xv_x+Tv_t-V,\quad \sigma^{\psi_1}=X\psi_{1x}+T\psi_{1t}-\Psi_1,\quad \sigma^{\psi_2}=X\psi_{2x}+T\psi_{2t}-\Psi_2,\\
&\sigma^{\phi_1}=X\phi_{1x}+T\phi_{1t}-\Phi_1,\quad \sigma^{\phi_2}=X\phi_{2x}+T\phi_{2t}-\Phi_2,\quad \sigma^p=Xp_x+Tp_t-P,
\end{aligned}\tag{17}$$

where $X, T, U, V, \Psi_1, \Psi_2, \Phi_1, \Phi_2$, and P are functions with respect to $\{x, t, u, v, \psi_1, \psi_2, \phi_1, \phi_2, p\}$, and ϵ is a small parameter.

Then substituting (17) into the linearized symmetry equations of the prolonged system

$$\begin{aligned}
&\sigma_t^u-\frac{1}{2}\sigma_{xxx}^u+3\sigma^u+3u\sigma_x^u-3\sigma_x^v=0,\quad \sigma_t^v+\sigma_{xxx}^v-3u\sigma_x^v-3\sigma^u v_x=0,\\
&\sigma_{xx}^{\psi_1}-\sigma^u\psi_1-u\sigma^{\psi_1}-\sigma^v\psi_2-v\sigma^{\psi_2}=0,\quad \sigma_{xx}^{\psi_2}-\sigma^u\psi_2-u\sigma_{\psi_2}-\sigma^{\psi_1}=0,\\
&\sigma_t^{\psi_1}-\frac{1}{2}\sigma_x^u\psi_1-\frac{1}{2}u_x\sigma^{\psi_1}+\sigma_x^v\psi_2+v_x\sigma^{\psi_2}+\sigma^u\psi_{1x}+u\sigma_x^{\psi_1}-2\sigma^v\psi_{2x}-2v\sigma_x^{\psi_2}=0,\\
&\sigma_t^{\psi_2}-\frac{1}{2}\sigma_x^u\psi_2-\frac{1}{2}u_x\sigma^{\psi_2}-2\sigma_x^{\psi_1}+\sigma^u\psi_{2x}+u\sigma_x^{\psi_2}=0,\quad \sigma_x^{\psi_1}-\sigma^{\phi_1}=0,\quad \sigma_x^{\psi_2}-\sigma^{\phi_2}=0,\\
&\sigma_x^p-2\sigma^{\psi_2}\psi_2=0,\quad \sigma_t^p-2\sigma^{\psi_2}\psi_2u-\psi_2^2\sigma^u-4\sigma^{\psi_1}\psi_2-4\psi_1\sigma^{\psi_2}+4\sigma^{\phi_2}\phi_2=0,
\end{aligned}\tag{18}$$

so collecting coefficients of the variables and their partial derivatives, and setting all to zero, we obtain a system of overdetermined linear equations of the infinitesimals $\{x, t, u, v, \psi_1, \psi_2, \phi_1, \phi_2, p\}$. By solving them, one can get

$$\begin{aligned} &X = c_1 x + c_2, \quad T = 3c_1 t + c_4, \quad U = -2c_3\psi_2\phi_2 - 2c_1 u, \quad V = c_3(\psi_1\phi_2 - \psi_2\phi_1) - 4c_1 v, \\ &\Psi_1 = -\frac{1}{2}c_3 p\psi_1 + c_5\psi_1, \quad \Psi_2 = -\frac{1}{2}c_3 p\psi_2 + (2c_1 + c_5)\psi_2, \quad \Phi_1 = -\frac{1}{2}c_3(\psi_1\psi_2^2 + p\phi_1) + (c_5 - c_1)\phi_1, \\ &\Phi_2 = -\frac{1}{2}c_3(\psi_2^3 + p\phi_2) + (c_1 + c_5)\phi_2, \quad P = -\frac{1}{2}c_3 p^2 + (5c_1 + 2c_5)p + c_6, \end{aligned} \tag{19}$$

where $c_i (i = 1 \dots 6)$ are six arbitrary constants. Especially, when $c_1 = c_2 = c_4 = c_5 = c_6 = 0$, the obtained symmetry is just Eq. (11), and when $c_3 = c_5 = c_6 = 0$, the related symmetry is only the general Lie point symmetry of (1).

To give more corresponding group invariant solutions, one need to solve the following characteristic equations:

$$\frac{dx}{X} = \frac{dt}{T} = \frac{du}{U} = \frac{dv}{V} = \frac{d\psi_1}{\Psi_1} = \frac{d\psi_2}{\Psi_2} = \frac{d\phi_1}{\Phi_1} = \frac{d\phi_2}{\Phi_2} = \frac{dp}{P}. \tag{20}$$

Next, several different similarity reductions arising from (20) are considered under the condition $c_3 \neq 0$ in detail.

Reduction 1. $c_1 \neq 0$. Without loss of generality, we assume $c_2 = c_4 = c_5 = 0$ and redefine the parameter c by $c^2 = \frac{25c_1^2 + 2c_3c_6}{36c_1^2}$ $(c \neq 0)$. By solving (20), we derive similarity solutions

$$\begin{aligned} u &= \frac{U(z)}{t^{\frac{2}{3}}} - \frac{c_3}{18c^2c_1^2t^{\frac{2}{3}}}\exp\left(-\frac{2}{3}P(z)\right)\Psi_2(z)(12cc_1\Phi_2(z)\tanh\Delta_1 + c_3\Psi_2^3(z)\operatorname{sech}^2\Delta_1), \\ v &= \frac{V(z)}{t^{\frac{4}{3}}} + \frac{c_3}{3cc_1t^{\frac{4}{3}}}\exp\left(-\frac{4}{3}P(z)\right)(\Psi_1(z)\Phi_2(z) - \Psi_2(z)\Phi_1(z))\tanh\Delta_1, \\ \psi_1 &= \frac{\Psi_1(z)}{t^{\frac{5}{6}}}\exp\left(-\frac{5}{6}P(z)\right)\operatorname{sech}\Delta_1, \quad \psi_2 = \frac{\Psi_2(z)}{t^{\frac{1}{6}}}\exp\left(-\frac{1}{6}P(z)\right)\operatorname{sech}\Delta_1, \\ \phi_1 &= \frac{1}{6cc_1t^{\frac{7}{6}}}\exp\left(-\frac{7}{6}P(z)\right)(6cc_1\Phi_1(z) - c_3\Psi_1(z)\Psi_2^2(z)\tanh\Delta_1)\operatorname{sech}\Delta_1, \\ \phi_2 &= \frac{1}{6cc_1t^{\frac{1}{2}}}\exp\left(-\frac{1}{2}P(z)\right)(6cc_1\Phi_2(z) - c_3\Psi_2^3(z)\tanh\Delta_1)\operatorname{sech}\Delta_1, \quad p = \frac{c_1}{c_3}(5 + 6c\tanh\Delta_1), \end{aligned} \tag{21}$$

with $\Delta_1 = c(\ln t + P(z))$, and the similarity variable $z = x/\sqrt[3]{t}$.

Here, $U(z), V(z), \Psi_1(z), \Psi_2(z), \Phi_1(z), \Phi_2(z), P(z)$, and z in (21) represent eight group invariants and substituting (21) into the prolonged system yields

$$\begin{aligned} U(z) &= \frac{\Psi_{2zz}(z)}{\Psi_2(z)} - \exp\left(-\frac{2}{3}P(z)\right)\frac{\Psi_1(z)}{\Psi_2(z)} - \frac{c_3}{9c_1c^2}\exp\left(-\frac{1}{3}P(z)\right)\Psi_2(z)\Psi_{2z}(z) \\ &\quad + \frac{c_3^2(1+12c^2)}{432c_1^2c^4}\exp\left(-\frac{2}{3}P(z)\right)\Psi_2^4(z), \\ V(z) &= \exp\left(-\frac{2}{3}P(z)\right)\left(\frac{\Psi_{1zz}(z)}{\Psi_2(z)} - \frac{\Psi_1(z)\Psi_{2zz}(z)}{\Psi_2^2(z)}\right) - \frac{c_3}{18c_1c^2}\exp(-P(z))(\Psi_1(z)\Psi_{2z}(z) \\ &\quad - \Psi_{1z}(z)\Psi_2(z)) + \exp\left(-\frac{4}{3}P(z)\right)\left(\frac{\Psi_1^2(z)}{\Psi_2^2(z)} + \frac{2c_3^2}{81c_1^2c^4}\Psi_1(z)\Psi_2^3(z)\right), \\ \Psi_1(z) &= \frac{cc_1\exp(P(z))Q(z)}{\sqrt{6c_1c_3\exp\left(\frac{1}{3}P(z)\right)P_z(z)}}, \quad \Psi_2(z) = \frac{c}{c_3}\sqrt{6c_1c_3\exp\left(\frac{1}{3}P(z)\right)P_z(z)}, \end{aligned} \tag{22}$$

$$\Phi_1(z) = \exp\left(\frac{1}{3}P(z)\right)\Psi_{1z}(z) - \frac{5c_3}{36c_1c^2}\Psi_1(z)(\Psi_2^2(z)), \quad \Phi_2(z) = \exp\left(\frac{1}{3}P(z)\right)\Psi_{2z}(z) - \frac{c_3}{36c_1c^2}(\Psi_2^3(z)),$$

where $P_z(z) \equiv P_1(z)$ and $Q(z)$ satisfy the ordinary differential equations

$$\begin{aligned}
&6P_1(z)P_{1zz}(z) - 9P_{1z}^2(z) + 4P_1^2(z)z - 12P_1(z) + 6P_1(z)Q(z) - 12c^2P_1^4(z) = 0,\\
&6P_1^2(z)Q_{zzz}(z) - 18P_1(z)P_{1z}(z)Q_{zz}(z) + 4P_1^2(z)Q_z(z)z + 18P_1(z)Q(z)Q_z(z) + 9P_{1z}^2(z)Q_z(z)\\
&\quad - 18P_1(z)Q_z(z) - 12P_{1z}(z)Q^2(z) + 18P_{1z}(z)Q(z) - 36c^2P_1^4(z)Q_z(z) = 0.
\end{aligned} \tag{23}$$

It appears naturally that when $P_1(z)$ and $Q(z)$ are solved from Eqs. (23), the explicit solutions of (1) would be immediately obtained through Eqs. (21) with Eqs. (22) and (23).

Reduction 2. $c_1 = 0$. Without loss of generality, let $c_4 = 1$ and redefine the parameter k by $k^2 = c_5^2 + \frac{1}{2}c_3c_6$ ($k \neq 0$). By solving (20), we derive similarity solutions

$$\begin{aligned}
&u = U(z) - \frac{2c_3}{k}\Psi_2(z)\Phi_2(z)\tanh\Delta_2 - \frac{c_3^2}{2k^2}\Psi_2^4(z)\mathrm{sech}^2\Delta_2,\\
&v = V(z) + \frac{c_3}{k}(\Psi_1(z)\Phi_2(z) - \Psi_2(z)\Phi_1(z))\tanh\Delta_2,\\
&\psi_1 = \Psi_1(z)\mathrm{sech}\Delta_2, \quad \psi_2 = \Psi_2(z)\mathrm{sech}\Delta_2, \quad \phi_1 = \left(\Phi_1(z) - \frac{c_3}{2k}\Psi_1(z)\Psi_2^2(z)\tanh\Delta_2\right)\mathrm{sech}\Delta_2,\\
&\phi_2 = \left(\Phi_2(z) - \frac{c_3}{2k}\Psi_2^3(z)\tanh\Delta_2\right)\mathrm{sech}\Delta_2, \quad p = \frac{2}{c_3}(c_5 + k\tanh\Delta_2),
\end{aligned} \tag{24}$$

with $\Delta_2 = k(t + P(z))$, and the similarity variable $z = x - c_2t$.

Substituting (24) into the prolonged system yields

$$\begin{aligned}
&U(z) = \frac{\Psi_{2zz}(z)}{\Psi_2(z)} - \frac{\Psi_1(z)}{\Psi_2(z)} + \frac{c_3^2}{k^2}\Psi_2^4(z),\\
&V(z) = \frac{\Psi_{1zz}(z)}{\Psi_2(z)} - \frac{\Psi_1(z)\Psi_{2zz}(z)}{\Psi_2^2(z)} + \frac{\Psi_1^2(z)}{\Psi_2^2(z)},\\
&\Phi_1(z) = \Psi_{1z}(z), \quad \Phi_2(z) = \Psi_{2z}(z),\\
&\Psi_1(z) = \frac{kQ(z)}{\sqrt{2c_3P_z(z)}}, \quad \Psi_2(z) = \frac{k\sqrt{2c_3P_z(z)}}{c_3},\\
&Q(z) = \frac{2}{3}(1 - c_2P_1(z) + k^2P_1^3(z)) + \frac{1}{2}\frac{P_{1z}^2(z)}{P_1(z)} - \frac{1}{3}P_{1zz}(z),
\end{aligned} \tag{25}$$

where $P_z(z) \equiv P_1(z)$ satisfy the ordinary differential equation

$$P_{1z}^2(z) = -2P_{1z}(z) + (4c_2 + 6)P_{1z}^2(z) + a_3P_{1z}^3(z) + 4c^2P_{1z}^4(z). \tag{26}$$

After summarizing the above formulas, the explicit solution of (1) would be immediately obtained. The dynamic behaviors are illustrated in Fig. 1 by solving Eq. (26) with a special case. This kind of solution can be easily applicable to the analysis of interesting physical phenomenon. In fact, there are full of the solitary waves and the cnoidal periodic waves in the real physics world.

4. Summary and discussion

In summary, the nonlocal symmetry of the DSSH system is obtained from the Lax pair, and infinitely many nonlocal symmetries are obtained by introducing the internal parameters. Then the nonlocal symmetry

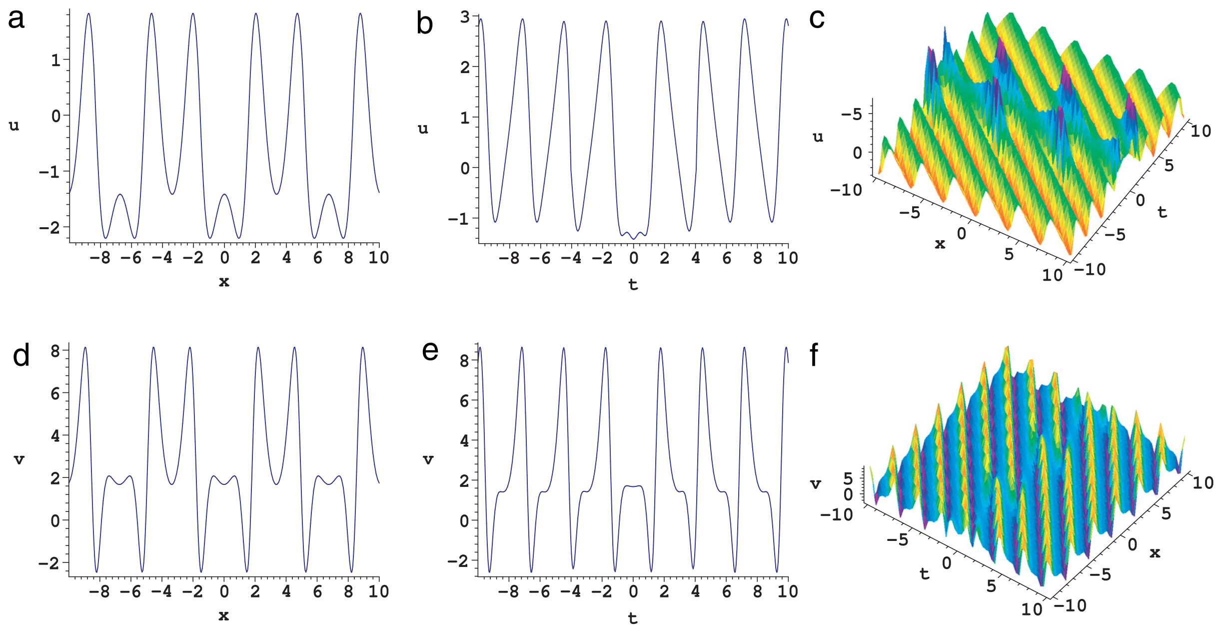

Fig. 1. The wave propagation plots of the DSSH system for solutions u and v with the solution of Eq. (26) are given as $P_1(z) = \frac{1}{\mu_0+\mu_1 sn^2(mz,n)}$ and the parameters $m = 1, n = \frac{1}{2}, \mu_0 = 1, \mu_1 = -\frac{1}{2}, c_2 = -\frac{5}{4}, k = 1$. (a) and (d) The wave propagation pattern of the wave along x axis at $t = 0$; (b) and (e) The wave propagation pattern of the wave along t axis at $x = 0$; (c) and (f) The three-dimensional plot of the corresponding solution.

is successfully localized to a prolonged system and the Schwartz form of the DSSH system reduced by the nonlocal symmetry from Lax pair is consistent with the truncated Painlevé expansion, which provides us a way to obtain the Schwartz form of integrable models. Meanwhile, the nonlocal symmetry is just related to the Möbious transformation of the Schwartz form. By using Lie point symmetry method, finite symmetry transformations and similarity reductions of the prolonged system are considered, several exact interaction solutions among solitons and other waves including periodic cnoidal waves, rational waves, and Painlevé waves are presented. These kinds of solutions can be easily applied to the analysis of many interesting physical phenomena, and this may provide us with a way to construct some new solutions for the integrable models with the known Lax pair. The details deserve further exploration in the future.

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