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Analysis of Incompressible Miscible Displacement in Porous Media by Characteristics Collocation Method

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Miscible displacement of one incompressible fluid by another in a porous medium is modelled by a coupled system of two partial differential equations. The pressure equation is elliptic, whereas the concentration equation is parabolic but normally convection-dominated. In this article, the collocation scheme is used to approximate the pressure equation and another characteristics collocation scheme to treat concentration equation. Existence and uniqueness of solutions of the algorithm are proved. Optimal order error estimate is demonstrated. © 2005 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 22: 797–814, 2006

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I. INTRODUCTION

Numerical simulation for incompressible flow in porous media is extremely important for developing oil fields reasonably and understanding fluid dynamics of oil and water under the ground. Two-phase incompressible flow is modelled by a coupled system of two partial differential equations, one elliptic partial differential equation for pressure of fluid and another convection-dominated diffusion equation for concentration. Let $\Omega = (0,1) \times (0,1)$ with the boundary $\partial\Omega$, $p(x,y,t)$ the pressure in the mixture, $\mathbf{u}(x,y,t)$ the Darcy velocity of the fluid, and $c(x,y,t)$ the relative concentration of the injected fluid. $k(x,y)$ and $\phi(x,y)$ are the permeability and the porosity of porous media, $\mu(c)$ is the viscosity of fluid, $D(x,y)$ is molecular dissipation coefficient, $a(c) = k(x,y)/\mu(c)$, $\bar{q}(t)$ and $\tilde{c}(t)$ etc. are just like the definitions of [1]. The mathematical

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controlling model for two-phase incompressible flow in porous media is given by

$$(a)\quad \nabla \cdot \mathbf{u} = -\nabla \cdot (a(c)\nabla p) = q(t), \qquad (x, y) \in \Omega,\ t \in (0, T],$$

$$(b)\quad \phi(x, y)\frac{\partial c}{\partial t} - \nabla \cdot (D(x, y)\nabla c) + \mathbf{u} \cdot \nabla c + \overline{q}(t)c(t) = \overline{q}(t)\overline{c}(t), \qquad (x, y) \in \Omega,\ t \in (0, T], \tag{1.1}$$

associated with boundary value conditions

$$(a)\quad \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \qquad t \in (0, T],$$

$$(b)\quad D\frac{\partial c}{\partial n}\bigg|_{\partial\Omega} = 0, \qquad t \in (0, T], \tag{1.2}$$

where $\mathbf{n}$ is a unit vector normal to $\partial\Omega$, and the initial value

$$c(x, y, 0) = c_0(x, y), \qquad (x, y) \in \Omega. \tag{1.3}$$

There hold some basic physical conditions:

$$\begin{aligned} &0 < D_* \le D(x, y) \le D^* < \infty, \\ &0 < a_* \le a(c) \le a^* < \infty, \\ &0 < \phi_* \le \phi(x, y) \le \phi^* < \infty, \end{aligned} \tag{1.4}$$

and the compatibility condition for $\bar{q}$

$$\int_\Omega \bar{q}(x, y, t)d\Omega = 0. \tag{1.5}$$

The collocation methods are widely used for solving practice problems in engineering due to its easiness of implementation and high-order accuracy; see, e.g., Douglas and Dupont in [2], Fernandes and Fairweather in [3], and Lu [4–6] for linear elliptic and parabolic problems with constant coefficients, Bialecki and Cai in [7] for orthogonal space collocation schemes for elliptic boundary value problems with variable coefficients in two space variable, and Bialecki and Fernandes in [8, 9] for orthogonal spline collocation LM, ADI methods, and ADI Crank-Nicolson methods for linear parabolic problems with variable coefficients, respectively.

The mathematical controlling model for two-phase incompressible flow in porous media is a strongly nonlinear coupling system of partial differential equations of two different types. Nonlinear terms introduce many difficulties for convergence analysis of algorithms. In the present article, we use different collocation technique to treat equations of different types, usual collocation method to solve elliptic equation for pressure, and characteristic collocation scheme to approximate convection-dominated diffusion equation for concentration. We develop some techniques to analyze convergence of collocation algorithm for this strongly nonlinear system and prove optimal order error estimate.

The organization of the rest of the article is as follows. In section II, we present the formulation of the characteristic collocation scheme for nonlinear system (1.1). In section III, we prove the existence and uniqueness of the solution of the characteristic collocation scheme. In section IV, we analyze convergent rate of the scheme defined in section II.

Throughout the article, the symbols K and ε will denote, respectively, a generic constant and a generic small positive constant independent of grid size of time and space partitions. We also assume that the problem is periodic in space, then the boundary condition (1.2) can be omitted.

II. FULLY DISCRETE CHARACTERISTIC COLLOCATION SCHEME

In this section, first, we give some basic notations and definition for collocation methods, which will be used in this article, and then we present the fully discrete characteristic collocation scheme for nonlinear system (1.1).

A. Notations and Definition for Collocation Methods

We make the partition of the domain Ω, which is quasi-uniform and equally spaced rectangular grid. The grid points are $(x_i, y_j), i = 0, 1 \cdots N_{xi} j = 0, 1 \cdots N_y$. Let

$$\delta_x : 0 = x_0 < x_1 < \cdots < x_{N_x} = 1, \qquad \delta_y : 0 = y_0 < y_1 < \cdots < y_{N_y} = 1$$

be grid points along x-direction and y-direction respectively, and

$$h_x = x_i - x_{i-1}, \qquad h_y = y_j - y_{j-1}, \qquad h = \max\{h_x, h_y\}$$

be grid size along x-direction and y-direction and maximum size of partition, respectively. Introduce the following notations:

$$\Omega_{ij} = (x_{i-1}, x_i) \times (y_{j-1}, y_j), \qquad I = [0, 1],$$
$$I_x^i = [x_{i-1}, x_i], \qquad I_y^j = [y_{j-1}, y_j],$$

for $i = 1, 2 \cdots N_x$ and $j = 1, 2 \cdots N_y$. Define function spaces as follows:

$$\mathcal{M}_1(3, \delta_x) = \left\{\upsilon \in C^1(I) | \upsilon \in P_3(I_x^i), i = 1 \cdots N_x\right\},$$
$$\mathcal{M}_1(3, \delta_y) = \left\{\upsilon \in C^1(I) | \upsilon \in P_3(I_y^j), j = 1 \cdots N_y\right\},$$

where P_3 denotes the set of polynomials of degree ≤ 3, and

$$\mathcal{M}_{1,P}(3, \delta_x) = \left\{\upsilon \in \mathcal{M}_1(3, \delta_x) : \upsilon(0) = \upsilon(1)\right\},$$
$$\mathcal{M}_{1,P}(3, \delta_y) = \left\{\upsilon \in \mathcal{M}_1(3, \delta_y) : \upsilon(0) = \upsilon(1)\right\},$$

then let $\mathcal{M}_1(3, \delta)$ and $\mathcal{M}_{1,P}(3, \delta)$ be the spaces of piecewise Hermite bicubics defined by

$$\mathcal{M}_1(3, \delta) = \mathcal{M}_1(3, \delta_x) \otimes \mathcal{M}_1(3, \delta y),$$

and

$$\mathcal{M}_{1,P}(3, \delta) = \mathcal{M}_{1,P}(3, \delta_x) \otimes \mathcal{M}_{1,P}(3, \delta_y).$$

Next, we take four Gauss points as collocation points in Ω_{ij} : (ξ_{ik}^x, ξ_{jl}^y), $k, l = 1, 2$,

$$\xi_{ik}^x = x_{i-1} + h_x \xi_k, \qquad \xi_{jl}^y = y_{j-1} + h_y \xi_l,$$

where

$$\xi_1 = (3 - \sqrt{3})/6, \qquad \xi_2 = (3 + \sqrt{3})/6.$$

Let T_{3,δ_x} and T_{3,δ_y} be the interpolation operators of piecewise Hermite bicubics of $\mathcal{M}_1(3,\delta_x)$ in x and $\mathcal{M}_1(3,\delta_y)$ in y, respectively, and $T_{3,\delta}$ be the interpolation operator of piecewise Hermite bicubics in $\mathcal{M}_1(3,\delta)$ on Ω, which may be defined by

$$T_{3,\delta}\upsilon = T_{3,\delta_x}T_{3,\delta_y}\upsilon = T_{3,\delta_y}T_{3,\delta_x}\upsilon,$$

for sufficiently smooth function υ.

Introduce the following summation notation:

$$\begin{aligned}
\langle u,\upsilon\rangle &= \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\langle u,\upsilon\rangle_{ij} = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{1}{4}h_xh_y\sum_{k,l=1}^{2}(u,\upsilon)(\xi_{ik}^x,\xi_{jl}^y),\\
\langle u,\upsilon\rangle_x &= \sum_{i=1}^{N_x}\langle u,\upsilon\rangle_{ix} = \sum_{i=1}^{N_x}\frac{h_x}{2}\sum_{k=1}^{2}(u\upsilon)(\xi_{ik}^x),\\
\langle u,\upsilon\rangle_y &= \sum_{j=1}^{N_y}\langle u,\upsilon\rangle_{jy} = \sum_{j=1}^{N_y}\frac{h_y}{2}\sum_{l=1}^{2}(u\upsilon)(\xi_{jl}^y),
\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}
\langle u,\upsilon\rangle &= \langle\langle u,\upsilon\rangle_x,1\,\rangle_y = \langle\langle u,\upsilon\rangle_y,1\rangle_x, \langle u,u\rangle = |||u|||^2,\\
|||u|||^2_{H_0^1(\Omega)} &= \int_0^1\langle Du_x,u_x\rangle_y\,dx + \int_0^1\langle Du_y,u_y\rangle_x\,dy, \qquad \forall u\in M_1(3,\delta),\\
|||u|||_E^2 &= \int_0^1\langle u_x,u_x\rangle_y\,dx + \int_0^1\langle u_y,u_y\rangle_x\,dy, \qquad \forall u\in M_1(3,\delta).
\end{aligned}\tag{2.2}$$

B. Fully Discrete Collocation Scheme

We use the following notation for the partition of time [0,**T**]:

$$\begin{aligned}
\Delta t_c &= \text{time step size for the concentration equation},\\
\Delta t_p^0 &= \text{first time step size for the pressure equation},\\
\Delta t_p &= \text{subsequent time step size for the pressure equation},\\
j &= \Delta t_p/\Delta t_c \in Z^+, \qquad j^0 = \Delta t_p^0/\Delta t_c \in Z^+,\\
t^n &= n\Delta t_c, \qquad t_m = \Delta t_p^0 + (m-1)\Delta t_p,
\end{aligned}$$

Where Z^+ is the set of positive integers. We note the fact that the variance of pressure with respect to time t is slower than the variance of concentration. So we use larger time step size to calculate the pressure. Also introduce some interpolating operators:

$$E\psi^n = \begin{cases}
\psi 0, & t^n \le t_1,\\
(1+\frac{\upsilon}{j^0})\psi_1 - \frac{\upsilon}{j^0}\psi_0, & t_1 \le t^n \le t_2,\ t^n = t_1 + \upsilon\Delta t_c,\\
(1+\frac{\upsilon}{j})\psi_m - \frac{\upsilon}{j}\psi_{m-1}, & t_m \le t^n \le t_{m+1}, \qquad t^n = t_m + \upsilon\Delta t_c.
\end{cases}\tag{2.3}$$

We consider the concentration equation. Let

$$\psi(x,y) = \left[\phi(x,y)^2 + |\mathbf{u}(x,y,t)|^2\right]^{1/2}$$

and the characteristic direction associated with the operator $\phi c_t + \mathbf{u}\cdot\nabla c$ is denoted by $s(x,y)$:

$$\frac{\partial}{\partial s(x,y)} = \frac{\phi}{\psi(x,y)}\frac{\partial}{\partial t} + \frac{u_1}{\psi(x,y)}\frac{\partial}{\partial x} + \frac{u_2}{\psi(x,y)}\frac{\partial}{\partial y}, \tag{2.4}$$

hence

$$\psi\frac{\partial c}{\partial s} = \phi\frac{\partial c}{\partial t} + \mathbf{u}\cdot\nabla c.$$

The equation (1.1) (b) can be put in the form

$$\psi\frac{\partial c}{\partial s} - \nabla\cdot(D\nabla c) + \overline{q}(t)c = \overline{q}(t)\overline{c}, \qquad (x,y)\in\Omega,\ t\in(0,T]. \tag{2.5}$$

In (2.5), we use a backward difference quotient for $\partial c/\partial s$ along the characteristic line:

$$\psi\frac{\partial c^{n+1}}{\partial s} \approx \psi\frac{c^{n+1}(x,y) - c^n(\breve{x},\breve{y})}{\Delta t_c[1+|\mathbf{u}|^2/\phi^2]^{\frac{1}{2}}} = \phi\frac{c^{n+1}-\breve{c}^n}{\Delta t_c}, \tag{2.6}$$

where

$$\breve{f}^n = f(\breve{x}^n,\breve{y}^n,t^n), \qquad f_{\mathrm{m}} = f(t_m), \qquad f^n = f(t^n), \tag{2.7}$$

with

$$\breve{x}^n = x - \frac{u_1^{n+1}}{\phi}\Delta t_c, \qquad \breve{y}^n = y - \frac{u_2^{n+1}}{\phi}\Delta t_c. \tag{2.8}$$

Then, we have the following discrete equation:

$$\phi\frac{c_h^{n+1}-\breve{c}_h^n}{\Delta t_c} - \nabla\cdot(D\nabla c_h^{n+1}) + \bar{q}^{n+1}(t)c_h^{n+1} = \bar{q}^{n+1}\bar{c}^{n+1}, \qquad n=0,1\cdots. \tag{2.9}$$

Now by using the interpolation operator $T_{3,\delta}$ and the Gauss points $\left\{(\xi_{ik}^x,\xi_{jl}^y), 1\le i\le N_x;\ 1\le j\le N_y; k,l=1,2\right\}$, we give the fully discrete characteristic collocation scheme.

Characteristic Collocation Scheme (CCS)

Seek C^n and P_m in the order.

$$C^0, P_0, C^1\cdots C^{j^0}, P_1, \qquad C^{j^0+1}\cdots C^{j^0+j}, P_2\cdots$$

such that

$$\begin{aligned}
&(a)\quad C^0 = T_{3,\delta}c_0(x,y),\\
&(b)\quad \left\{-\nabla\cdot(a(C_m)\nabla P_m) - q_m\right\}(\xi_{ik}^x,\xi_{jl}^y) = 0,\\
&(c)\quad \left\{\phi\frac{C^{n+1}-\hat{C}^n}{\Delta t_c} - \nabla\cdot(D\nabla C^{n+1}) - (\overline{c}^{n+1}-C^{n+1})\overline{q}^{n+1}\right\}(\xi_{ik}^x,\xi_{jl}^y) = 0,
\end{aligned} \tag{2.10}$$

where

$$\hat{f}^n = f(\hat{x}^n, \hat{y}^n, t^n), \qquad f_m = f(t_m), \qquad f^n = f(t^n), \tag{2.11}$$

and

$$\mathbf{U}_m = -a(C_m)\nabla P_m, \tag{2.12}$$

with

$$\hat{x}^n = x - \frac{EU_1^{n+1}}{\phi}\Delta t_c, \qquad \hat{y}^n = y - \frac{EU_2^{n+1}}{\phi}\Delta t_c, \tag{2.13}$$

for $1 \le i \le N_x, 1 \le j \le N_y, k, l = 1, 2$, and $n, m \ge 0$.

In next section, we will analyze existence and convergence of the solution of the characteristic collocation scheme.

III. EXISTENCE OF SOLUTION OF CCS

In this section, we analyze the existence and uniqueness of the solution of the characteristic collocation scheme. To prove the existence and uniqueness of the solution of the characteristic collocation scheme in this section and convergence in next section, we require the following lemmas. At first we list some basic results in [2, 10].

Lemma 3.1. *Let $e = \upsilon - T_{3,\delta_x}\upsilon$, then there exists constant $K > 0$ such that*

$$\begin{aligned}
&(1) \quad \langle e^l, e^l\rangle_x \le Kh_x^{2(4-l)} \cdot \sum_{i=1}^{N_x} \int_{x_{i-1}}^{x_i} \sum_{\alpha\le 4} \left(\frac{\partial^\alpha \upsilon}{\partial x^\alpha}\right)^2 dx, \qquad l = 0, 1,\\
&(2) \quad \langle e_{xx}, e_{xx}\rangle_x \le Kh_x^6 \cdot \sum_{i=1}^{N_x} \int_{x_{i-1}}^{x_i} \sum_{\alpha\le 5} \left(\frac{\partial^\alpha \upsilon}{\partial x^\alpha}\right)^2 dx,\\
&(3) \quad |\langle e_x, 1\rangle_x|^2 \le Kh_x^9 \cdot \sum_{i=1}^{N_x} \int_{x_{i-1}}^{x_i} \sum_{\alpha\le 5} \left(\frac{\partial^\alpha \upsilon}{\partial x^\alpha}\right)^2 dx,\\
&(4) \quad |\langle e_{xx}, 1\rangle_x|^2 \le Kh_x^9 \cdot \sum_{i=1}^{N_x} \int_{x_{i-1}}^{x_i} \sum_{\alpha\le 6} \left(\frac{\partial^\alpha \upsilon}{\partial x^\alpha}\right)^2 dx.
\end{aligned} \tag{3.1}$$

There is the same conclusions in y direction.

Lemma 3.2. *There exists constant $K \ge 0$ such that for sufficiently smooth function υ*

$$\begin{aligned}
\|\upsilon - T_{3,\delta}\upsilon\|_{L^2(\Omega)} &\le Kh^4 \left(\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \|\upsilon^{(4)}\|_{L^2(\Omega_{ij})}\right)^{1/2},\\
\|\upsilon - T_{3,\delta}\upsilon_t\|_{L^2(\Omega)} &\le Kh^4 \left(\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \|\upsilon_t^{(4)}\|_{L^2(\Omega_{ij})}\right)^{1/2}.
\end{aligned} \tag{3.2}$$

Lemma 3.3. *Let* $v \in \mathcal{M}_1(3,\delta).v \equiv 0$ *if and only if the following condition:*

$$\begin{aligned} v(\xi_{ik}^x,0) = v(\xi_{ik}^x,1) = v(0,\xi_{jl}^y) = v(1,\xi_{jl}^y) = v(0,0) = v(0,1) = v(1,0) \\ = v(1,1) = v(\xi_{ik}^x,\xi_{jl}^y) = 0, \end{aligned} \tag{3.3}$$

hold for $1 \le i \le N_x, 1 \le j \le N_y$ *and* $k,l = 1,2$.

Proof. We can see the lemma 3.3 in [2]. ∎

Lemma 3.4. *Assume that* $D(x,y)$ *is sufficiently smooth. There exist constants* $0 < K_* \le K^*$ *and* $h_0 > 0$ *such that for any* h *in* $(0,h_0]$ *and each* $v \in \mathcal{M}_{1,P}(3,\delta)$

$$K_*\langle -\Delta v, v\rangle \le -\langle \nabla\cdot(D\nabla v), v\rangle \le K^*\langle -\Delta v, v\rangle. \tag{3.4}$$

Proof. The Peano representation of the remainder in the two-point Gauss-Legendre quadrature and Leibnitz's formula (see Theorem 4.2 in [7]) reads

$$\left\langle -\frac{\partial}{\partial x}\left(D\frac{\partial v}{\partial x}\right)(\cdot,\eta_{jl}), v(\cdot,\eta_{jl})\right\rangle_x = I_1(D,v,\eta_{jl}) + I_2(D,v,\eta_{jl}),$$

where $\mathcal{K}$ is Peano kernel and

$$\begin{aligned} I_1(D,v,\eta_{jl}) &= \int_0^1 \left[D\left(\frac{\partial v}{\partial x}\right)^2\right](x,\eta_{jl})dx + 4\sum_{k=1}^{N_x}(h_x)^4\int_{I_k^x}\left[D\left(\frac{\partial^3 v}{\partial x^3}\right)^2\right](x,\eta_{jl})\mathcal{K}\left(\frac{x-x_{k-1}}{h_x}\right)d_x \\ &= I_3(D,v,\eta_{jl}) + I_4(D,v,\eta_{jl}), \end{aligned}$$

and

$$I_2(D,v,\eta_{jl}) = \sum_{i=1}^{5}\sum_{\substack{m+n=6-i\\ 0\le m,n\le 3}}\alpha_{m,n}^i\sum_{k=1}^{N_x}(h_x)^4 \times \int_{I_k^x}\left[\frac{\partial^i D}{\partial x^i}\frac{\partial^m v}{\partial x^m}\frac{\partial^n v}{\partial x^n}\right](x,\eta_{jl})\mathcal{K}\left(\frac{x-x_{k-1}}{h_x}\right)dx,$$

constants α_{mn}^i are independent of h and symmetrical $\alpha_{mn}^i = \alpha_{nm}^i$, and

$$0 \le \mathcal{K}(\beta) = \frac{1}{24}\left\{(1-\beta)^4 - 2\left[(\xi_1-\beta)_+^3 + (\xi_2-\beta)_+^3\right]\right\} \le K, \qquad \beta \in [0,1].$$

Since $I_2(1,v,\eta_{jl}) = 0$, we see that

$$D_*\left\langle -\frac{\partial^2 v}{\partial x^2}(\cdot,\eta_{jl}), v(\cdot,\eta_{jl})\right\rangle_x \le I_1(D,v,\eta_{jl}), \quad D_* \in \min_{(x,y)\in\overline{\Omega}} D(x,y). \tag{3.5}$$

On the other hand, by using the Cauchy-Schwarz inequality in $L^2(I_k^x)$, we have

$$|I_2(D,v,\eta_{jl})| \le K\sum_{i=1}^{5}\sum_{\substack{m+n=6-i\\ 0\le m,n\le 3}}\sum_{k=1}^{N_x}(h_x)^4\|D\|_{W^{i,\infty}(I_k^x)}\|v(\cdot,\eta_{jl})\|_{H^m(I_k^x)}\|v(\cdot,\eta_{jl})\|_{H^n(I_k^x)}. \tag{3.6}$$

Hence, by using the inverse inequality

$$\|\upsilon\|_{H^i(I_k^x)} \le K h_x^{m-i} \|\upsilon\|_{H^m(I_k^x)}, \qquad 0 \le m \le i \le 3, \qquad \upsilon \in P_3,$$

and $\|\upsilon\|_{L^\infty(0,1)} \le K\|\upsilon\|_{H^1(0,1)}$, for $\upsilon \in \mathcal{M}_{1,P}(3,\delta_x)$, we get

$$0 \le I_4(D,\upsilon,\eta_{jl}) \le K D^* \left\| \frac{\partial \upsilon}{\partial x}(\cdot,\eta_{jl}) \right\|^2_{L^2(0,1)}, \qquad D^* = \max_{(x,y)\in\overline{\Omega}} D(x,y). \tag{3.7}$$

and

$$|I_2(D,\nu,\eta_{jl})| \le K \left(\sum_{i=1}^{5} \max_{l\le k\le N_x} (h_x)^{\min(4,i)} \|D\|_{W^{i,\infty}(I_k^x)} \right) \|\upsilon(\cdot,\eta_{jl})\|^2_{H^1((0,1))},$$

such that

$$|I_2(D,\nu,\eta_{jl})| \le K K_1^x h_x \left\| \frac{\partial \upsilon}{\partial x}(\cdot,\eta_{jl}) \right\|^2_{L^2(0,1)}, \tag{3.8}$$

where

$$K_1^x = \sum_{i=0}^{4} \max_{l\le k\le N_x} (h_x)^{\min(3,i)} \|D\|_{W^{i+1,\infty}(I_k^x)}.$$

Further, lemma 3.3 of [2] implies that

$$|I_2| \le K K_1^x h_x \left\langle -\frac{\partial^2 \upsilon}{\partial x^2}(\cdot,\eta_{jl}), \upsilon(\cdot,\eta_{jl}) \right\rangle_x,$$

and

$$|I_4| \le K D^* \left\langle -\frac{\partial^2 \upsilon}{\partial x^2}(\cdot,\eta_{jl}), \upsilon(\cdot,\eta_{jl}) \right\rangle_x.$$

Putting above estimates together, we have

$$(D_* - K K_1^x h_x) \left\langle -\frac{\partial^2 \upsilon}{\partial x^2}, \upsilon \right\rangle \le \left\langle -\frac{\partial}{\partial x}\left(D \frac{\partial \upsilon}{\partial x} \right), \upsilon \right\rangle \le (D^* + K K_1^x h_x + K D^*) \left\langle -\frac{\partial^2 \upsilon}{\partial x^2}, \upsilon \right\rangle. \tag{3.9}$$

For $\left\langle -\frac{\partial}{\partial y}\left(D \frac{\partial \upsilon}{\partial y} \right), \upsilon \right\rangle$, we have the similar estimate:

$$(D_* - K K_1^x h_x) \left\langle -\frac{\partial^2 \upsilon}{\partial y^2}, \upsilon \right\rangle \le \left\langle -\frac{\partial}{\partial y}\left(D \frac{\partial \upsilon}{\partial y} \right), \upsilon \right\rangle \le (D^* + K K_1^x h_x + K D^*) \left\langle -\frac{\partial^2 \upsilon}{\partial y^2}, \upsilon \right\rangle. \tag{3.10}$$

Let $K_* = D_* - K K_1 h$ and $K^* = D^* + K K_1 h + K D^*$. For sufficient small h, K_* is positive. (3.4) is proved. This ends the proof of lemma 3.4. ∎

The following conclusion in [2] and [6] will be needed:

$$|||\upsilon|||_E^2 \le -\langle \Delta\upsilon, \upsilon\rangle \le K|||\upsilon|||_E^2, \qquad \upsilon \in \mathcal{M}_{1,P}(3,\delta), \tag{3.11}$$

and

$$\|\upsilon\|_{H^1(\Omega)}^2 \le K\left\{\langle \upsilon, \upsilon\rangle + |||\upsilon|||_{H_0^1(\Omega)}^2\right\}; \qquad \upsilon \in \mathcal{M}_{1,}(3,\delta), \tag{3.12}$$

Then (2.2) of [3] implies that

$$\|\upsilon\|_{L^2(\Omega)} \le |||\upsilon||| \le K_1\|\upsilon\|_{L^2(\Omega)}, \qquad \upsilon \in \mathcal{M}_{1,}(3,\delta). \tag{3.13}$$

Lemma 3.5. *Under the same conditions as in Lemma 3.4, there exists constant* $0 < C_* \le C^*$ *such that*

$$C_*|||\upsilon|||_{H_0^1(\Omega)}^2 \le \langle -\nabla\cdot(D\nabla\upsilon), \upsilon\rangle \le C^*|||\upsilon|||_{H_0^1(\Omega)}^2, \qquad \forall\upsilon \in \mathcal{M}_{1,P}(3,\delta). \tag{3.14}$$

Proof. By using (1.4) and (2.2), we obtain

$$D_*|||\upsilon|||_E^2 \le |||\upsilon|||_{H_0^1(\Omega)}^2 \le D^*|||\upsilon|||_E^2, \qquad \upsilon \in \mathcal{M}_{1,P}(3,\delta).$$

By using (3.11) and (3.4), we have

$$\begin{aligned}\frac{K_*}{D^*}|||\upsilon|||_{H_0^1(\Omega)}^2 \le K_*|||\upsilon|||_E^2 &\le K_*\langle -\Delta\upsilon, \upsilon\rangle \\ &\le -\langle \nabla\cdot(D\nabla\upsilon), \upsilon\rangle \le K^*\langle -\Delta\upsilon, \upsilon\rangle \\ &\le K^*K|||\upsilon|||_E^2 \le \frac{K^*K}{D_*}|||\upsilon|||_{H_0^1(\Omega)}^2, \qquad \upsilon \in \mathcal{M}_{1,P}(3,\delta).\end{aligned}$$

Let $C_* = K_*/D^*$ and $C^* = K^*K/D_*$ leads to (3.14). The proof of Lemma 3.5 is completed. ■

Now we can show the existence and uniqueness of the numerical solution. First, we discuss the concentration equation. [2] show that (2.10)(c) may be rewritten as the discrete Galerkin method given by

$$\left\langle \phi\frac{C^{n+1} - \hat{C}^n}{\Delta t_c} - \nabla\cdot(D\nabla C^{n+1}) - (\bar{c}^{n+1} - C^{n+1})\bar{q}^{n+1}, Z\right\rangle = 0, \qquad \forall Z \in \mathcal{M}_{1,P}(3,\delta). \tag{3.15}$$

Let $\{Z_i\}_{i=1}^{4N_xN_y}$ be a basis for $m_{1,P}(3,\delta)$ then for any $C(x,y) \in \mathcal{M}_{1,P}(3,\delta)$ there exists

$$C^n(x,y) = \sum_{i=1}^{4N_xN_y} \beta_i^n Z_i(x,y). \tag{3.16}$$

(3.15) is rewritten as the matrix form:

$$E\beta^{n+1} + D\beta^n = S, \tag{3.17}$$

where

$$E = (E_{ij})_{4N_xN_y\times 4N_xN_y}, \qquad E_{ij} = \phi\langle Z_j, Z_i\rangle - \Delta t_c\langle\nabla\cdot(D\nabla Z_j), Z_i\rangle + \Delta t_c\bar{q}^{n+1}\langle Z_j, Z_i\rangle$$
$$D = (D_{ij})_{4N_xN_y\times 4N_xN_y}, \qquad D_{ij} = -\phi\langle\hat{Z}, Z_i\rangle,$$
$$S = (S_i)_{4N_xN_y}, \qquad S_i = \Delta t_c\langle\bar{c}^{n+1}\bar{q}^{n+1}, Z_i\rangle.$$

It is clear that $A = (a_{ij})_{4N_xN_y\times 4N_xN_y}$, where $a_{ij} = \langle Z_j, Z_i\rangle$, is nonsingular. Assume that $A\tau = 0$. Further, E also is nonsingular for sufficiently small Δt_c so that (3.15) has one unique solution and is equivalent to (2.10)(c).

The pressure equation (2.10)(b) is equivalent to the corresponding discrete Galerkin method

$$\langle -\nabla\cdot(a(C_m)\nabla P_m) - q_m, \chi\rangle = 0, \qquad \forall\chi\in\mathcal{M}_{1,P}(3,\delta), \tag{3.18}$$

here C_m has been known. Similar to (3.9) and (3.10), we have estimate

$$\langle -\nabla\cdot(a(C_m)\nabla\upsilon), \upsilon\rangle \geq (a_* - KK_2h)\langle -\Delta\upsilon, \upsilon\rangle, \tag{3.19}$$

where K_2 is given by

$$K_2 = \sum_{i=0}^{4}\max_{1\leq k_1\leq N_x, 1\leq k_2\leq N_y}(h_x+h_y)^{\min(3,i)}\|a(C_m)\|_{W^{i+1,\infty}(I^x_{k_1}\times I^y_{k_2})}. \tag{3.20}$$

Noting that $\partial^k_x C_m \equiv 0$ and $\partial^k_y C_m \equiv 0$ for $k > 3$, we can easily show that

$$(a)\quad \|a(C_m)\|_{W^{2,\infty}(I^x_{k_1}\times I^y_{k_2})} \leq K\big((h_x+h_y)^{-1}\|a'\|_{L^\infty}\|C_m\|_{W^{1,\infty}} + \|a''\|_{L^\infty}\|C_m\|^2_{W^{1,\infty}}\big),$$

$$(b)\quad \|a(C_m)\|_{W^{3,\infty}(I^x_{k_1}\times I^y_{k_2})} \leq K\big((h_x+h_y)^{-2}\|a'\|_{L^\infty}\|C_m\|_{W^{1,\infty}} + (h_x+h_y)^{-1}\|a''\|_{L^\infty}\|C_m\|^2_{W^{1,\infty}} + \|a'''\|_{L^\infty}\|C_m\|^3_{W^{1,\infty}}\big),$$

$$(c)\quad \|a(C_m)\|_{W^{4,\infty}(I^x_{k_1}\times I^y_{k_2})} \leq K\big((h_x+h_y)^{-2}\|a''\|_{L^\infty}\|C_m\|^2_{W^{1,\infty}} + (h_x+h_y)^{-1}(\|a'''\|_{L^\infty} + \|a^{(3)}\|_{L^\infty})\|C_m\|^3_{W^{1,\infty}} + \|a^{(4)}\|_{L^\infty}\|C_m\|^4_{W^{1,\infty}}\big),$$

$$(d)\quad \|a(C_m)\|_{W^{5,\infty}(I^x_{k_1}\times I^y_{k_2})} \leq K\big((h_x+h_y)^{-3}\|a''\|_{L^\infty}\|C_m\|^2_{W^{1,\infty}} + (h_x+h_y)^{-2}(\|a'''\|_{L^\infty} + \|a^{(4)}\|_{L^\infty})\|C_m\|^3_{W^{1,\infty}} + (h_x+h_y)^{-1}\|a^{(4)}\|_{L^\infty}\|C_m\|^4_{W^{1,\infty}} + \|a^{(5)}\|_{L^\infty}\|C_m\|^5_{W^{1,\infty}}\big).$$

Substituting estimates above into (3.20) leads to

$$K_2 = K_2(\|a\|_{W^{5,\infty}}, \|C_m\|_{W^{1,\infty}}), \tag{3.21}$$

where $\|a\|_{W^{5,\infty}}$ is the norm of a(z) with respect to z. If $a\in W^{5,\infty}$ and $\|C_m\|_{W^{1,\infty}}$ is uniformly bounded, then K_2 also is uniformly bounded. In next section, we will prove that $\|C_m\|_{W^{1,\infty}}$ is uniformly bounded in practical calculation. Under this condition, we know that there exists constants $h_0 > 0$ and $a_0 > 0$ such that for any $0 < h \leq h_0$,

$$a_* - KK_2h \geq a_0 > 0. \tag{3.22}$$

This implies that the system (3.18) has one unique solution for a sufficiently small h.

IV. CONVERGENCE ANALYSIS

In this section, we will analyze error estimate in L^2-norm. We assume that the exact solution of the system (1.1) satisfies the regularity (R):

$$c \in L^\infty(H^6) \cap L^\infty(W_\infty^2) \cap H^1(W_\infty^2) \cap H^2(H^1),$$
$$p \in L^\infty(H^6) \cap H^1(H^6) \cap L^\infty(W_\infty^1) \cap H^2(H^1).$$

We have the following error estimate of optimal order accuracy.

Theorem 4.1. *Suppose (R) and $r = 3$ hold, and that the space and time discretizations satisfy the relations*

$$\Delta t_c = O(h^3), \qquad \Delta t_p^0 = O((\Delta t_c)^{2/3}), \qquad \Delta t_p = O((\Delta t_c)^{1/2}). \tag{4.1}$$

Then there exists a constant K such that, for h sufficiently small,

$$\max_{0\le n\le\left[\frac{T}{\Delta tc}\right]} \|c^n - C^n\|^2 + \sum_{m=0}^{T/\Delta t_p} \|p_m - P_m\|^2 \Delta t_c \le K(\Delta t_c^2 + h^8). \tag{4.2}$$

To prove the Theorem 4.1, we require some lemmas. Let $\tilde{c} = T_{3,\delta}c$, $\tilde{p} = T_{3,\delta}p$ and

$$\xi = c - \tilde{c}, \qquad \zeta = C - \tilde{c}, \qquad \theta = p - \tilde{p}, \qquad \eta = P - \tilde{p}.$$

Lemma 4.1. *There holds the following estimate:*

$$|\langle (D\xi_x^{n+1})_x, \zeta^{n+1}\rangle| \le \varepsilon\left\{(\zeta_x^{n+1}, \zeta^{n+1}) + \langle \zeta^{n+1}, \zeta^{n+1}\rangle\right\} + Kh^8\|c^{n+1}\|_{H^6}^2, \tag{4.3}$$

for $\varepsilon > 0$. There is the same conclusion in y direction.

Proof. It is clear that

$$\langle (D\xi_x^{n+1})_x, \zeta^{n+1}\rangle_x = \langle D_x\xi_x^{n+1}, \zeta^{n+1}\rangle_x + \langle D\xi_{xx}^{n+1}, \zeta^{n+1}\rangle_x. \tag{4.4}$$

We estimate the first term of the right-hand side of (4.4). Let $\check{\zeta}_i^{n+1} = h_x^{-1}\langle \zeta^{n+1}, 1\rangle_i$. It follows from (2.1) that

$$\langle \zeta^{n+1}, 1\rangle_i^2 \le K\frac{h_x^2}{4}\left\{(\zeta^{n+1}(\xi_{i1}^x))^2 + (\zeta^{n+1}(\xi_{i2}^x))^2\right\} \le Kh_x\langle \zeta^{n+1}, \zeta^{n+1}\rangle_i. \tag{4.5}$$

Thus

$$|\check{\zeta}_i^{n+1}| \le Kh_x^{-\frac{1}{2}}|||\zeta^{n+1}|||_i. \tag{4.6}$$

By using $\check{\zeta}_i^{n+1}$ we have

$$|\langle D_x\xi_x^{n+1}, \zeta^{n+1}\rangle_i| \le |\langle D_x\xi_x^{n+1}, \zeta^{n+1} - \check{\zeta}_i^{n+1}\rangle_i| + |\langle D_x\xi_x^{n+1}, \check{\zeta}_i^{n+1}\rangle_i| = S_1 + S_2.$$

Using of (3.1) and Poincáre inequality, we obtain

$$|S_1| \le K \max\left\{|D_x(\xi_{i1}^x)|, |D_x(\xi_{i2}^x)|\right\} |||\xi_x^{n+1}|||_i \cdot |||\zeta^{n+1} - \check{\zeta}_i^{n+1}|||_i$$

$$\le K h_x^4 \max\left\{|D_x(\xi_{i1}^x)|, |D_x(\xi_{i2}^x)|\right\} \left(\int_{x_{i-1}}^{x_i} \sum_{\alpha\le 4} \left(\frac{\partial^\alpha c^{n+1}}{\partial x^\alpha}\right)^2 dx\right)^{\frac{1}{2}} \cdot \|\zeta_x^{n+1}\|_{L^2(I_i)}$$

$$\le \varepsilon(\zeta_x^{n+1}, \zeta_x^{n+1})_i + K h_x^8 \int_{x_{i-1}}^{x_i} \sum_{\alpha\le 4} \left(\frac{\partial^\alpha c^{n+1}}{\partial x^\alpha}\right)^2 dx. \quad (4.7)$$

By use of (3.1), (4.5), and (4.6), we obtain

$$|S_2| \le K \max\left\{|D_x(\xi_{i1}^x)|, |D_x(\xi_{i2}^x)|\right\} |\langle \xi_x^{n+1}, 1\rangle_i| \cdot |\check{\zeta}_i^{n+1}|$$

$$\le K h_x^4 \max\left\{|D_x(\xi_{i1}^x)|, |D_x(\xi_{i2}^x)|\right\} \left(\int_{x_{i-1}}^{x_i} \sum_{\alpha\le 5} \left(\frac{\partial^\alpha c^{n+1}}{\partial x^\alpha}\right)^2 dx\right)^{\frac{1}{2}} \cdot |||\zeta^{n+1}|||_i$$

$$\le \varepsilon\langle \zeta_x^{n+1}, \zeta^{n+1}\rangle_i, + K h_x^8 \int_{x_{i-1}}^{x_i} \sum_{\alpha\le 4} \left(\frac{\partial^\alpha c^{n+1}}{\partial x^\alpha}\right)^2 dx \quad (4.8)$$

Next we estimate the second term of (4.4)

$$|\langle D\xi_{xx}^{n+1}, \zeta^{n+1}\rangle_i| \le |\langle D\xi_{xx}^{n+1}, \zeta^{n+1} - \zeta_i^{n+1}\rangle_i| + |\langle D\xi_{xx}^{n+1}, \check{\zeta}_i^{n+1}\rangle_i| = S_1' + S_2'.$$

Similar to (4.7) and (4.8), we have

$$|S_1'| \le K \max\left\{|D(\xi_{i1}^x)|, |D(\xi_{i2}^x)|\right\} |||\xi_{xx}^{n+1}|||_i \cdot |||\zeta^{n+1} - \check{\zeta}_i^{n+1}|||_i$$

$$\le K h_x^4 \max\left\{|D(\xi_{i1}^x)|, |D(\xi_{i2}^x)|\right\} \left(\int_{x_{i-1}}^{x_i} \sum_{\alpha\le 5} \left(\frac{\partial^\alpha c^{n+1}}{\partial x^\alpha}\right)^2 dx\right)^{\frac{1}{2}} \cdot \|\zeta_x^{n+1}\|_{L^2(I_i)}$$

$$\le \varepsilon(\zeta_x^{n+1}, \zeta_x^{n+1})_i + K h_x^8 \int_{x_{i-1}}^{x_i} \sum_{\alpha\le 5} \left(\frac{\partial^\alpha c^{n+1}}{\partial x^\alpha}\right)^2 dx. \quad (4.9)$$

and

$$|S_2'| \le K \max\left\{|D(\xi_{i1}^x)|, |D(\xi_{i2}^x)|\right\} |\langle \xi_{xx}^{n+1}, 1\rangle_i| \cdot |\check{\zeta}_i^{n+1}|$$

$$\le K h_x^4 \max\left\{|D(\xi_{i1}^x)|, |D(\xi_{i2}^x)|\right\} \left(\int_{x_{i-1}}^{x_i} \sum_{\alpha\le 6} \left(\frac{\partial^\alpha c^{n+1}}{\partial x^\alpha}\right)^2 dx\right)^{\frac{1}{2}} \cdot |||\zeta^{n+1}|||_i$$

$$\le \varepsilon\langle \zeta^{n+1}, \zeta^{n+1}\rangle_i + K h_x^8 \int_{x_{i-1}}^{x_i} \sum_{\alpha\le 6} \left(\frac{\partial^\alpha c^{n+1}}{\partial x^\alpha}\right)^2 dx. \quad (4.10)$$

By summing over i, we get

$$|\langle (D\xi_x^{n+1})_x, \zeta^{n+1}\rangle_x| = \left|\sum_{i=1}^{N_x}\langle (D\xi_x^{n+1})_x, \zeta^{n+1}\rangle_i\right|$$
$$= \left|\sum_{i=1}^{N_x}\left[\langle D_x\xi_x^{n+1}, \zeta^{n+1}\rangle_i + \langle D\xi_{xx}^{n+1}, \zeta^{n+1}\rangle_i\right]\right|$$
$$\le \varepsilon\left\{(\zeta_x^{n+1}, \zeta_x^{n+1})_x + \langle \zeta^{n+1}, \zeta^{n+1}\rangle_x\right\} + Kh_x^8\int_0^1\sum_{\alpha\le 6}\left(\frac{\partial^\alpha c^{n+1}}{\partial x^\alpha}\right)^2 dx. \quad (4.11)$$

Summing (4.11) over x leads to (4.3). This ends the proof of Lemma 4.1. ∎

Lemma 4.2. *There holds the following estimate:*

$$\left|\left\langle (a(c_m)\theta_x^{n+1})_x, \eta^{n+1}\right\rangle\right| \le \varepsilon\left\{\langle \eta^{n+1}, \eta^{n+1}\rangle + (\eta_x^{n+1}, \eta_x^{n+1})\right\} + Kh^8\|p^{n+1}\|_{H^6}^2, \quad (4.12)$$

for $\varepsilon > 0$. There is the same conclusion in y direction.

Lemma 4.3. *There holds the following estimate:*

$$\left|\left\langle\left[(a(c_m) - a(\tilde{c}_m))\tilde{p}_x^{n+i}\right]_x, \eta^{n+1}\right\rangle_x\right| \le \varepsilon\left\{\langle \eta^{n+1}, \eta^{n+1}\rangle_x + (\eta_x^{n+1}, \eta_x^{n+1})_x\right\} + Kh^8\|c_m\|_{H^5}^2, \quad (4.13)$$

for $\varepsilon > 0$. There is the same conclusion in y direction.

Proof. It is clear that

$$\left\langle\left[(a(c_m) - a(\tilde{c}_m))\tilde{p}_x^{n+1}\right]_x, \eta^{n+1}\right\rangle_x = \left\langle (a(c_m) - a(\tilde{c}_m))_x, \tilde{p}_x^{n+1}\eta^{n+1}\right\rangle_x$$
$$+ \left\langle (a(c_m) - a(\tilde{c}_m))\tilde{p}_{xx}^{n+1}, \eta^{n+1}\right\rangle_x. \quad (4.14)$$

Let $(\tilde{p}_x^{n+1}\eta^{n+1})_i^* = h_x^{-1}\langle \tilde{p}_x^{n+1}\eta^{n+1}, 1\rangle_i$. We have

$$\left|\left\langle (a(c_m) - a(\tilde{c}_m))_x, \tilde{p}_x^{n+1}\eta^{n+1}\right\rangle_i\right| + \left|\left\langle (a(c_m) - a(\tilde{c}_m))\tilde{p}_{xx}^{n+1}, \eta^{n+1}\right\rangle_i\right|$$
$$\le \left|\left\langle (a(c_m) - a(\tilde{c}_m))_x, \tilde{p}_x^{n+1}\eta^{n+1} - (\tilde{p}_x^{n+1}\eta^{n+1})_i^*\right\rangle_i\right|$$
$$+ \left|\langle (a(c_m) - a(\tilde{c}_m))_x, 1\rangle_i(\tilde{p}_x^{n+1}\eta^{n+1})_i^*\right| + \left|\left\langle (a(c_m) - a(\tilde{c}_m))\tilde{p}_{xx}^{n+1}, \eta^{n+1}\right\rangle_i\right|$$
$$\le h_x^4\sum_{\alpha\le 5}\|\partial_x^\alpha c_m\|_{L^2(I_i)}\left(|||\eta^{n+1}|||_i + \|\eta_x^{n+1}\|_i\right)$$
$$\le \varepsilon\left(|||\eta^{n+1}|||_i^2 + \|\eta_x^{n+1}\|_i^2\right) + Kh_x^8\sum_{\alpha\le 5}\|\partial_x^\alpha c_m\|_{L^2(I_i)}^2. \quad (4.15)$$

Summing over i and y leads to (4.13). The proof of Lemma 4.3 is completed. ∎

Lemma 4.4. *Let ∂_s be direction derivative given by (2.4). There holds the following estimate:*

$$\left|\left|\left|\psi\frac{\partial c^{n+1}}{\partial s} - \phi\frac{c^{n+1} - \breve{c}^n}{\Delta t_c}\right|\right|\right|^2 \le K\Delta t_c\int_{t_n}^{t_{n+1}}\cdot\int_\Omega\left|\frac{\partial^2 c}{\partial s^2}\right|^2 d\Omega dt. \quad (4.16)$$

Proof. Noting that

$$\psi\frac{\partial c^{n+1}}{\partial s}=\phi\frac{\partial c^{n+1}}{\partial t}+\mathbf{u}^{n+1}\cdot\nabla c^{n+1}.$$

Along the tangent to the characteristic, we see that [11, 12]

$$\psi\frac{\partial c^{n+1}}{\partial s}-\phi\frac{c^{n+1}-\breve{c}^{n}}{\Delta t_c}=\frac{\phi}{\Delta t_c}\int_{(\breve{x},\breve{y},t^n)}^{(x,y,t^{n+1})}\sqrt{(x(s)-\breve{x})^2+(y(s)-\breve{y})^2+(t(s)-t^n)^2}\;\frac{\partial^2 c}{\partial s^2}ds. \tag{4.17}$$

so by use of (2.1), we obtain

$$\left\langle\psi\frac{\partial c^{n+1}}{\partial s}-\phi\frac{c^{n+1}-\breve{c}^{n}}{\Delta t_c},\psi\frac{\partial c^{n+1}}{\partial s}-\phi\frac{c^{n+1}-\breve{c}^{n}}{\Delta t_c}\right\rangle$$
$$=\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\frac{1}{4}h_xh_y\sum_{k,l=1}^{2}\left\{\left(\frac{\phi}{\Delta t_c}\int_{(\breve{x},\breve{y},t^n)}^{(x,y,t^{n+1})}\sqrt{(x-\breve{x})^2+(y-\breve{y})^2+(t-t^n)^2}\frac{\partial^2 c}{\partial s^2}ds\right)(\xi_{ik}^x,\xi_{jl}^y)\right\}^2.$$

Let E_{ij} be the plane from $(\breve{\xi}_{ik}^x,\breve{\xi}_{jl}^y,t^n)$ to $(\xi_{ik}^x,\xi_{jl}^y,t^{n+1})$ along the characteristic direction, then

$$\left|\!\left|\!\left|\psi\frac{\partial c^{n+1}}{\partial\tau}-\phi\frac{c^{n+1}-\breve{c}^{n}}{\Delta t_c}\right|\!\right|\!\right|^2$$
$$\le Kh^2\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\sum_{k,l=1}^{2}\max_{(x,y)\in E_{ij}}\left|\frac{\partial^2 c}{\partial\tau^2}\right|^2\left\{\frac{\phi}{\Delta t_c}\cdot\left(\frac{\psi\Delta t_c}{\phi}\right)\int_{(\breve{\xi}_{ik},\breve{\xi}_{jl},t^n)}^{(\xi_{ik},\xi_{jl},t^{n+1})}ds\right\}^2$$
$$\le K\Delta t_c\int_{t_n}^{t_{n+1}}\int_{\Omega}\left|\frac{\partial^2 c}{\partial s^2}\right|^2 d\Omega dt.$$

The proof of Lemma 4.4 ends. ∎

Proof of Theorem 4.1. The proof is divided two steps. At first step, we analyze the pressure equation and then, at the second step, we analyze the concentration equation.

At the first step, we consider the pressure equation. Subtracting (3.18) from the Galerkin formulation of (1.1)(*a*), we obtain the error equation of pressure

$$-\langle\nabla\cdot(a(C_m)\nabla\eta_m),\chi\rangle=-\langle\nabla\cdot(a(c_m)\nabla\theta_m),\chi\rangle-\langle\nabla\cdot[(a(c_m)-a(C_m))\nabla\tilde{p}_m],\chi\rangle. \tag{4.18}$$

By choosing the test function $\chi=\eta_m$ in (4.18) and writing the right-hand terms of (4.18) as $T_i',i=1,2$ in turn, we have

$$-\langle\nabla\cdot(a(C_m)\nabla\eta_m),\eta_m\rangle=T_1'+T_2'. \tag{4.19}$$

In order to estimate (4.19), we need an induction hypothesis. We assume that

$$\|C_m\|_{W^{1,\infty}}\le\|c\|_{L^\infty(W^{\infty,1})}+1,\qquad 0\le m\le l-1. \tag{4.20}$$

We shall check that (4.20) is true at the end of the proof.

It follows from (3.19) and (3.22) that

$$-\langle \nabla \cdot (a(C_m)\nabla \eta_m)\eta_m \rangle \geq a_0 \langle -\Delta \eta_m, \eta_m \rangle \geq a_0 \|\nabla \eta_m\|^2, \tag{4.21}$$

for sufficiently small h. From Lemma 4.2 and Lemma 4.3, we derive that

$$|T_1'| = |\langle \nabla \cdot (a(c_m)\nabla \theta_m), \eta_m \rangle| \leq \varepsilon |||\nabla \eta_m|||^2 + Kh^8 \|p_m\|_{H^6}^2, \tag{4.22}$$

and

$$\begin{aligned} |T_2'| &= \left|\left\langle \nabla \cdot \left[(a(c_m) - a(C_m))\nabla \tilde{p}_m\right], \eta_m \right\rangle\right| \\ &\leq \left|\left\langle \nabla \cdot \left[(a(c_m) - a(\bar{c}_m))\nabla \tilde{p}_m\right], \eta_m \right\rangle\right| + \left|\left\langle \nabla \cdot \left[(a(\bar{c}_m) - a(C_m))\nabla \tilde{p}_m\right], \eta_m \right\rangle\right| \\ &\leq K\left(\|\zeta_m\|_1^2 + h^8 \|c_m\|_{H^5}^2\right) + \varepsilon |||\eta_m|||^2. \end{aligned} \tag{4.23}$$

Substituting (4.22), (4.23), and (4.21) into (4.18) leads to

$$a_0 \|\nabla \eta_m\|^2 \leq K\left\{\|\zeta_m\|_1^2 + h^8\left(\|p_m\|_{H^6}^2 + \|c_m\|_{H^5}^2\right)\right\} + \varepsilon |||\nabla \eta_m|||^2. \tag{4.24}$$

Taking $\varepsilon < a_0$, we get the estimate

$$|||\nabla \eta_m||| \leq K\left\{\|\zeta_m\|_1 + h^4\left(\|p_m\|_{H^6} + \|c_m\|_{H^5}\right)\right\}. \tag{4.25}$$

At the second step, we consider the concentration equation. Subtracting (3.15) from the discrete Galerkin scheme of (1.1)(b), we obtain

$$\begin{aligned} \left\langle \phi \frac{\zeta^{n+1} - \zeta^n}{\Delta t_c} - \nabla \cdot (D\nabla \zeta^{n+1}), Z \right\rangle &= \left\langle \phi \frac{\partial c^{n+1}}{\partial t} + \mathbf{u}^{n+1} \cdot \nabla c^{n+1} - \phi \frac{c^{n+1} - \check{c}^n}{\Delta t_c}, Z \right\rangle \\ &\quad - \left\langle \phi \frac{\check{c}^n - \hat{c}^n}{\Delta t_c}, Z \right\rangle - \langle \nabla \cdot (D\nabla \xi^{n+1}), Z \rangle + \left\langle \phi \frac{\xi^{n+1} - \hat{\xi}^n}{\Delta t_c}, Z \right\rangle - \left\langle \phi \frac{\zeta^n - \hat{\zeta}^n}{\Delta t_c}, Z \right\rangle \\ &\quad + \langle \bar{q}^{n+1} \xi^{n+1}, Z \rangle - \langle \bar{q}^{n+1} \zeta^{n+1}, Z \rangle, \qquad \forall Z \in \mathcal{M}_{1,P}(3, \delta). \end{aligned} \tag{4.26}$$

To obtain L^2 estimate for ζ, we choose $Z = \zeta^{n+1}$ as test function in (4.26), and then denote the resulting right-hand side terms by $T_1, T_2, \cdots, T_7$. It follows from lemma 4.4 that

$$|T_1| \leq K \left\| \psi \frac{\partial c^{n+1}}{\partial s} - \phi \frac{c^{n+1} - \check{c}^n}{\Delta t_c} \right\| \cdot |||\zeta^{n+1}||| \leq K \Delta t_c \int_{t_n}^{t_{n+1}} \int_\Omega \left| \frac{\partial^2 c}{\partial s^2} \right|^2 d\Omega dt + \varepsilon |||\zeta^{n+1}|||^2. \tag{4.27}$$

Noting that

$$
\begin{aligned}
|T_2| &= |\langle \nabla c \cdot (\mathbf{u}^{n+1} - E\mathbf{U}^{n+1}), \zeta^{n+1}\rangle| \\
&\le |\langle \nabla c \cdot (\mathbf{u}^{n+1} - E\mathbf{u}^{n+1}), \zeta^{n+1}\rangle| + K\left\{\left|\left\langle \nabla c \cdot \left(1+\frac{\nu}{j}\right)(\mathbf{u}_{m+1} - \mathbf{U}_{m+1}), \zeta^{n+1}\right\rangle\right|\right. \\
&\qquad \left. + \left|\left\langle \nabla c \cdot \left(\frac{\nu}{j}\right)(\mathbf{u}_m - \mathbf{U}_m), \zeta^{n+1}\right\rangle\right|\right\} \\
&\le K|\langle \nabla c \cdot (\mathbf{u}^{n+1} - E\mathbf{u}^{n+1}), \zeta^{n+1}\rangle| + \left\{\left|\left\langle \nabla c \cdot \left(1+\frac{\nu}{j}\right)[a(c_{m+1})\nabla\theta_{m+1} - a(C_{m+1})\nabla\eta_{m+1}\right.\right.\right. \\
&\qquad \left.\left.\left. + (a(c_{m+1}) - a(C_{m+1}))\nabla\tilde{p}_{m+1}], \zeta^{n+1}\right\rangle\right| \\
&\qquad + \left.\left|\left\langle \nabla c \cdot \left(\frac{\nu}{j}\right)[a(c_m)\nabla\theta_m - a(C_m)\nabla\eta_m + (a(c_m) - a(C_m))\nabla\tilde{p}_m], \zeta^{n+1}\right\rangle\right|\right\},
\end{aligned}
$$

We have by use of (4.1), Lemma 3.1 and Lemma 4.1

$$
|T_2| \le K\left(\|\zeta_m\|^2 + \|\zeta_{m+1}\|^2 + \|\zeta^{n+1}\|^2 + h^8\|c_m\|_{H^5}^2\right) + \varepsilon\left(\|\zeta^{n+1}\|_1^2 + |||\nabla\eta_m|||\right). \tag{4.28}
$$

It follows from Lemma 4.1 that

$$
|T_3| = \left|\left\langle \nabla\cdot(D\nabla\xi^{n+1}), \zeta^{n+1}\right\rangle\right| \le \varepsilon\left(|||\zeta^{n+1}|||^2 + |||\nabla\zeta^{n+1}|||^2\right) + Kh^8\|c^{n+1}\|_{H^6(\Omega)}^2. \tag{4.29}
$$

For T_4, it follows that

$$
|T_4| = \left|\left\langle \phi\frac{\xi^{n+1} - \hat{\xi}^n}{\Delta t_c}, \zeta^{n+1}\right\rangle\right| \le K\left\{\left|\left\langle \phi\frac{\xi^{n+1} - \xi^n}{\Delta t_c}, \zeta^{n+1}\right\rangle\right| + \left|\left\langle \phi\frac{\xi^n - \hat{\xi}^n}{\Delta t_c}, \zeta^{n+1}\right\rangle\right|\right\}.
$$

It follows from the Taylor expansion, Cauchy inequality and (3.1) that

$$
\left|\left\langle \phi\frac{\xi^{n+1} - \xi^n}{\Delta t_c}, \zeta^{n+1}\right\rangle\right| \le \varepsilon|||\zeta^{n+1}|||^2 + K|||\xi_t^{n+1}|||^2 \le \varepsilon|||\zeta^{n+1}|||^2 + Kh^8\|c_t^{n+1}\|_{H^4(\Omega)}^2. \tag{4.30}
$$

By using two-dimensional Taylor expansion, (2.3), the induction hypotheses (4.20), Lemma 3.1, and Lemma 4.1, we derive that for sufficiently small Δt_c:

$$
\begin{aligned}
\left|\left\langle \phi\frac{\xi^n - \hat{\xi}^n}{\Delta t_c}, \zeta^{n+1}\right\rangle\right| &\le K\left\{|\langle EU_1^{n+1}\xi_x^n, \zeta^{n+1}\rangle| + |\langle EU_2^{n+1}\xi_y^n, \zeta^{n+1}\rangle| + \Delta t_c\|c^n\|_{H^2}|||\zeta^{n+1}|||\right\} \\
&\le K\left\{|\langle E(-a(C^{n+1})\nabla\eta^{n+1})\xi_x^n, \zeta^{n+1}\rangle| + |\langle E(-a(C^{n+1})\nabla n^{n+1})\xi_y^n, \zeta^{n+1}\rangle|\right. \\
&\quad + |\langle E(-a(C^{n+1})\nabla\tilde{p}^{n+1})\xi_x^n, \zeta^{n+1}\rangle| + |\langle E(-a(C^{n+1})\nabla\tilde{p}^{n+1})\xi_y^n, \zeta^{n+1}\rangle| \\
&\qquad \left. + \Delta t_c\|c^n\|_{H^2}|||\zeta^{n+1}|||\right\} \\
&\le \varepsilon\left(|||\zeta^{n+1}|||^2 + \|\nabla\zeta^{n+1}\|^2 + \|\nabla\eta_m\|^2 + \|\nabla\eta_{m+1}\|^2\right) \\
&\qquad + K\left(h^8\|c^n\|_{H^5(\Omega)}^2 + \Delta t_c^2\|c^n\|_{H^2}^2 + \|\zeta^{n+1}\|^2\right).
\end{aligned} \tag{4.31}
$$

Thus we obtain the estimate of T_4 from (4.30) and (4.31)

$$|T_4| \le \varepsilon\big(|||\zeta^{n+1}|||^2 + \|\nabla\zeta^{n+1}\|^2 + +\|\nabla\eta_m\|^2 + \|\nabla\eta_{m+1}\|^2\big) + K\big(h^8\|c^n\|^2_{H^5(\Omega)} + h^8\|c_t^{n+1}\|^2_{H^4(\Omega)} + \Delta t_c^2 + \|\zeta^{n+1}\|^2\big). \quad (4.32)$$

Moreoever, it follows from (3.13), the induction hypotheses (4.20) that

$$|T_5| \le \left|\left|\left| \frac{\zeta^n - \hat{\zeta}^n}{\Delta t_c} \right|\right|\right| \cdot |||\zeta^{n+1}||| \le \varepsilon \left|\left|\left| \frac{\zeta^n - \hat{\zeta}^n}{\Delta t_c} \right|\right|\right|^2 + K\|\zeta^{n+1}\|^2 \le \varepsilon\|\nabla\zeta^n\|^2 + K\|\zeta^{n+1}\|^2. \quad (4.33)$$

From (1.4), Cauchy inequality and (3.1), we derive that

$$|T_6| + |T_7| \le K|||\zeta^{n+1}|||^2 + \varepsilon|||\zeta^{n+1}|||^2 \le Kh^8\|c^{n+1}\|^2_{H^4(\Omega)} + \varepsilon|||\zeta^{n+1}|||^2. \quad (4.34)$$

Substituting (3.13), (4.25) and (4.27)–(4.34) into the right-hand side of (4.26), we have

$$\left\langle \phi\frac{\zeta^{n+1} - \zeta^n}{\Delta t_c}, \zeta^{n+1} \right\rangle - \langle \nabla\cdot(D\nabla\zeta^{n+1}), \zeta^{n+1}\rangle \le \varepsilon\big(\|\zeta^{n+1}\|_1^2 + \|\zeta_m\|_1^2 + \|\zeta_{m+1}\|_1^2\big) + K\big\{\Delta t_c^2h^2 + \Delta t_c^2 + h^8 + \|\zeta_m\|^2 + \|\zeta_{m+1}\|^2 + \|\zeta^{n+1}\|^2\big\}. \quad (4.35)$$

By use of the inequality $\frac{1}{2}(a^2 - b^2) \le a(a-b)$, we see that the left-hand side of (4.35)

$$\frac{1}{2\Delta t_c}\big\{\langle\phi\zeta^{n+1}, \zeta^{n+1}\rangle - \langle\phi\zeta^n, \zeta^n\rangle\big\} - \langle\nabla\cdot(D\nabla\zeta^{n+1}), \zeta^{n+1}\rangle \le \left\langle \phi\frac{\zeta^{n+1} - \zeta^n}{\Delta t_c}, \zeta^{n+1}\right\rangle - \langle\nabla\cdot(D\nabla\zeta^{n+1}), \zeta^{n+1}\rangle. \quad (4.36)$$

By using (4.35), (4.36), and (3.14), we now have

$$\langle\phi\zeta^{n+1}, \zeta^{n+1}\rangle - \langle\phi\zeta^n, \zeta^n\rangle + 2\Delta t_c C_*|||\zeta^{n+1}|||^2_{H_0^1(\Omega)} \le K\Delta t_c\big\{\Delta t_c^2h^2 + \Delta t_c^2 + h^8 + \|\zeta^{n+1}\|^2 + \|\zeta_m\|^2 + \|\zeta_{m+1}\|^2\big\}. \quad (4.37)$$

Summing (4.37) in time n, we have

$$\langle\phi\zeta^{n+1}, \zeta^{n+1}\rangle + C_*\sum_{k=0}^{n}|||\zeta^{k+1}|||^2_{H_0^1(\Omega)}\Delta t_c \le K\left\{\Delta t_c^2 + h^8 + \sum_{k=0}^{n}\Delta t_c\|\zeta^k\|^2\right\}. \quad (4.38)$$

Further, applying the Gronwall lemma to (4.38) leads to

$$\max_{0\le k\le n}\|\zeta^{k+1}\|^2 + C_*\sum_{k=0}^{n}|||\zeta^{k+1}|||^2_{H_0^1(\Omega)}\Delta t_c \le K\{\Delta t_c^2 + h^8\}. \quad (4.39)$$

Combining (4.39) with (4.25) shows that

$$\sum_{m=0}^{n}\|\nabla\eta_m\|^2\Delta t_c \le K\{\Delta t_c^2 + h^8\}. \quad (4.40)$$

Finally, we shall check the induction hypotheses (4.20). By using the inverse property of finite element spaces, we see

$$\|C_1\|_{w^1_\infty} \le \|c\|_{L^\infty(W^{1,\infty})} + \|\zeta_l\|_{W^1_\infty} \le K + Kh^{-2}\|\zeta_l\|. \tag{4.41}$$

Applying (4.39) to (4.41) and noting $\Delta t_c = o(h)$, we have

$$\|C_1\|_{W^1_\infty} \le \|c\|_{L^\infty(W^{1,\infty})} + Kh \le \|c\|_{L^\infty(W^{1,\infty})} + 1,$$

for sufficiently small h. The proof of Theorem 4.1 is completed. ∎

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