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Adaptive optimal control approximation for solving a fourth-order elliptic variational inequality☆

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ABSTRACT

An optimal control approach is proposed to solve the fourth-order elliptic variational inequality with curvature obstacle. It is proved that the variational inequality is equivalent to the constrained optimal control problem. The finite element approximation of the optimal control problem is constructed and the a priori error estimates and the equivalent a posteriori error estimators are derived. Some numerical experiments are performed to confirm a priori error estimates and demonstrate the effectiveness of the a posteriori estimators.

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1. Introduction

Fourth-order elliptic variational inequalities with obstacles have been widely studied, such as variational inequalities with curvature obstacle, displacement obstacle, and so on. There have been many researches for solving these variational inequalities. For example, Noor and Al-Said used the finite difference method in [1,2], and they developed a new cubic spline method for computing the approximate solution of a system of fourth-order boundary value problems associated with an obstacle in [3]. Glowinski, Lions and Tremolieres used penalty and relaxation methods in [4,5], Momania, Moadia and Noor used the decomposition method in [6], Shi and Wang used non-conforming finite element methods in [7–11] to solve these fourth order variational inequalities with obstacles. Also, Brézis and Stampacchia studied the regularity of a fourth order variational inequality with curvature obstacle with two type of boundary conditions in [12]. Up to now, it is still a difficult and interesting problem to solve a fourth-order variational inequality effectively.

The purpose of this article is to develop a new approach to solve a fourth-order elliptic variational inequality with curvature obstacle. The idea is to use the adaptive optimal control approach to obtain the solution indirectly. To this end, we translate the fourth-order variational inequality with curvature obstacle into the equivalent form, which contains two Poisson equations and a 2rd order variational inequality. The optimality condition we obtained in this paper is equivalent to the mixed variational form deduced by Deng and Shen in [13]. In [13], they have the priori error estimates for $\|y - y_h\|_{H^1(\Omega)}$ and $\|u - u_h\|_{L^2(\Omega)}$ with the order $h^{1-\varepsilon}$, $\forall \varepsilon > 0$, but in our paper, we obtained the h^1 order convergence for the above two error estimates.

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The paper is organized as follows: The notations used throughout the paper are introduced in Section 2, together with the equivalence proof between the fourth-order variational inequality and the constrained optimal control problem, and also the optimality conditions are given. In Section 3, we present the finite element approximation of the control problem. In Section 4, we give the priori error estimate of y. Equivalent a posteriori error estimates with H^1 norm and L^2 norm are deduced in Section 5. In Section 6, we give two numerical experiments to demonstrate our error estimates developed in Sections 4 and 5.

2. Fourth-order variational inequality and its equivalent form

Let Ω be a convex domain in R^2 with the Lipschitz boundary $\partial\Omega$. In this paper we adopt the standard notation $W^{m,q}(\Omega)$ for the Sobolev spaces on Ω with norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and seminorm $|\cdot|_{W^{m,q}(\Omega)}$. We set $W_0^{m,q}(\Omega) \equiv \{w \in W^{m,q}(\Omega) : w|_{\partial\Omega} = 0\}$ and denote $W^{m,2}(\Omega)$ $(W_0^{m,2}(\Omega))$ by $H^m(\Omega)$ $(H_0^m(\Omega))$. In addition, c or C denotes a general positive constant independent of h.

2.1. Fourth-order variational inequality

Let $K = \{y \in H^2(\Omega) \cap H_0^1(\Omega) : \Delta y \le 0 \text{ in } \Omega\}$. Here, the condition satisfied in K is called the curvature obstacle. We consider the following fourth-order variational inequality problem: *Find y in K such that*

$$\int_\Omega \Delta y \Delta(w - y) \ge \int_\Omega f(w - y), \quad \forall w \in K \tag{2.1}$$

where $f \in L^2(\Omega)$ is a given function.

In engineering, the variational inequality problem (2.1) describes the curvature obstacle problem. In general it is very difficult to solve the fourth-order variational inequality problem. In the next subsection, we translate the variational inequality problem (2.1) into an equivalent optimal control problem, which is governed by the second order PDE, so that it can be solved easily by using known-well adaptive finite element methods.

2.2. Equivalent optimal control problem

Define a convex set K' of the form:

$$K' = \{u \in L^2(\Omega);\ u \ge 0 \text{ (a.e.) in } \Omega\}.$$

It is clear that K' is closed in $L^2(\Omega)$. We formulate the optimal control problem equivalent to the fourth-order variational inequality problem (2.1) in the following equivalent theorem.

Theorem 2.1. *The problem (2.1) is equivalent to the following optimal control problem:*

$$\begin{cases} \min\limits_{u\in K'} \left\{ J(y,u) = \dfrac{1}{2}\displaystyle\int_\Omega u^2 - \int_\Omega fy \right\}, \\ s.t.\ -\Delta y = u \quad in\ \Omega,\ y = 0\ on\ \partial\Omega. \end{cases} \tag{2.2}$$

Proof. It is clear that the minimized problem (2.2) may be represented by

$$\min_{y\in K} \left\{ J(y) = \frac{1}{2}\int_\Omega (\Delta y)^2 - \int_\Omega fy \right\}. \tag{2.3}$$

Obviously, J is convex. Suppose y is the solution of the minimized problem (2.3). From [5], we know the following conclusion:

$$J(y) = \min_{y\in K} J(y) \Longleftrightarrow J'(y)(w - y) \ge 0, \quad \forall w \in K.$$

Since

$$J'(y)(w - y) = \int_\Omega \Delta y \Delta(w - y) - \int_\Omega f(w - y) \ge 0, \quad \forall w \in K,$$

we deduce that $J'(y)(w - y) \ge 0$ for each $w \in K$ is equivalent to (2.1). Then Theorem 2.1 is proved. □

Theorem 2.1 shows that one could solve the optimal control problem (2.2) instead of solving the variational inequality problem (2.1). In the following parts we study how to solve the optimal control problem (2.2). To this end, we give a weak formula for the state equation. Let the state space $V = H_0^1(\Omega)$ and the control space $U = L^2(\Omega)$. Set

$$a(y, w) = \int_\Omega \nabla y \cdot \nabla w \quad \forall y, w \in V; \qquad (f, g) = \int_\Omega fg \quad \forall f,\ g \in U.$$

Therefore, the control problem (2.2) can be restated as the following:

$$(\text{OCP}): \quad \begin{cases} \min\limits_{u\in K'} \left\{ J(y,u) = \dfrac{1}{2}\displaystyle\int_\Omega u^2 - \int_\Omega fy \right\}, \\ a(y,w) = (u,w), \quad \forall w \in H_0^1(\Omega). \end{cases} \tag{2.4}$$

It is clear that the control problem (OCP) has a unique solution $(y,u) \in V \times U$.

Theorem 2.2 (*Optimality Condition*)**.** *A pair* (y,u) *is the solution of* (OCP) *if and only if there is a co-state* $p \in V$ *such that the triplet* $(y,p,u) \in V \times V \times U$ *satisfies the following optimality conditions:*

$$(\text{OCP–OPT}): \begin{cases} a(y,w) = (u,w), \quad \forall w \in V, \\ a(q,p) = (f,q), \quad \forall q \in V, \\ (u-p, v-u) \geq 0, \quad \forall v \in K'. \end{cases} \tag{2.5}$$

From (OCP–OPT) and [14] we know that

$$u = \max\{0,\ p\}. \tag{2.6}$$

The optimality condition (OCP–OPT) provides another way to approach the fourth-order variational inequality (2.1). The (OCP–OPT) has at least three advantages: one is that it is the second-order problem so as to be easily solved by using usual conforming finite elements; another is that the variational inequality in (OCP–OPT) is easily solved directly by (2.6). But it is very difficult to bring a curvature obstacle to finite element approximation. The third advantage, also most important in computation, is the system is separated, i.e.,

$$(\text{OCP–OPT}') \begin{cases} \text{(a) } a(q,p) = (f,q), \quad \forall q \in V, \\ \text{(b) } u = \max\{0,p\}, \\ \text{(c) } a(y,w) = (u,w), \quad \forall w \in V, \end{cases} \tag{2.7}$$

in which p is independent of y so that it can be solved independently from (2.7)(a), then u is obtained from (2.7)(b) and finally y is solved by (2.7)(c).

By use of the regularity theory of the partial differential equations of the second order elliptic type, we can directly obtain the following the regularity theorem.

Theorem 2.3 (*Regularity [15]*)**.** *Assume that* Ω *is a convex domain or with smooth boundary. Then* $u \in W^{1,\infty}(\Omega)$, $y \in H^2(\Omega) \bigcap H_0^1(\Omega)$ *and* $p \in H^2(\Omega) \bigcap H_0^1(\Omega)$. *There holds a priori estimate*

$$\begin{aligned} &\text{(a) } \|p\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}, \\ &\text{(b) } \|u\|_{W^{1,s}(\Omega)} \leq \|p\|_{W^{1,s}(\Omega)}, \quad 1 \leq s < \infty, \\ &\text{(c) } \|y\|_{W^{2,s}(\Omega)} \leq C\|u\|_{L^s(\Omega)}, \quad 1 \leq s \leq \infty. \end{aligned} \tag{2.8}$$

3. Finite element approximation

Let us consider the finite element approximation of the control problem (OCP). Here we consider only n-simplex elements, as they are widely used in engineering applications. Also we consider only conforming finite elements. For simplicity, we assume that Ω is a polygonal domain.

Let $T^h = \bigcup \tau$ be a partitioning of Ω into disjoint regular n-simplices τ, in which each element has at most one face on $\partial\Omega$, and $\bar{\tau}$ and $\bar{\tau}'$ have either only one common vertex or a whole edge or face if τ and $\tau' \in T^h$. Associated with T^h is a finite dimensional subspace S^h of $C(\bar{\Omega}^h)$, such that $\chi|_\tau$ are linear functions for each $\chi \in S^h$ and each $\tau \in T^h$. Let $V^h = S^h \cap H_0^1(\Omega)$. It is easy to see that $V^h \subset V$.

Let $T_U^h = \bigcup \tau_U$ be another partition of Ω such that $\bar{\Omega} = \bigcup_{\tau_U \in T_U^h} \bar{\tau}_U$. Assume that $\bar{\tau}_U$ and $\bar{\tau}_U'$ have at most either only one common vertex or a whole edge or face if τ_U and $\tau_U' \in T_U^h$. Associated with T_U^h is another finite dimensional subspace W_U^h of $L^2(\Omega)$, such that $\chi|_{\tau_U}$ are constants for each $\chi \in W_U^h$ and each $\tau_U \in T_U^h$. Let $U^h = W_U^h$, and $K'^h = (U^h \cap K')$.

Let $h_\tau(h_{\tau_U})$ denote the maximum diameter of the element $\tau(\tau_U)$ in $T^h(T_U^h)$.

Then a possible finite element approximation of (OCP), which we shall label $(\text{OCP})^h$, is as follows:

$$(\text{OCP})^h: \quad \begin{cases} \min\limits_{u_h \in K'^h} \dfrac{1}{2}\displaystyle\int_\Omega u_h^2 - \int_\Omega fy_h \\ a(y_h, w_h) = (u_h, w_h), \quad \forall w_h \in V^h. \end{cases} \tag{3.1}$$

The discretized control problem $(\text{OCP})^h$ has a unique solution $(y_h, u_h) \in V^h \times U^h$. Similarly as given in [16] or [14], we have the following Theorem 3.1.

Theorem 3.1. *A pair $(y_h, u_h) \in V^h \times U^h$ is the solution of* (OCP)^h *if and only if there is a co-state $p_h \in V^h$ such that the triplet (y_h, p_h, u_h) satisfies the following optimality conditions labeled as* (OCP–OPT)^h*:*

$$(\text{OCP–OPT})^h : \begin{cases} a(y_h, w_h) = (u_h, w_h), & \forall w_h \in V_h, \\ a(q_h, p_h) = (f, q_h), & \forall q_h \in V_h, \\ (u_h - p_h, v_h - u_h) \geq 0, & \forall v_h \in K'^h. \end{cases} \tag{3.2}$$

Since U^h is the space of piecewise constant elements, hence

$$u_h = \max\{0, \mathcal{P}_h p_h\}, \tag{3.3}$$

where $\mathcal{P}_h$ is the L^2 project from $L^2(\Omega)$ onto U^h such that

$$\mathcal{P}_h v|_{\tau_U} = \bar{v}|_\tau = \frac{1}{|\tau_U|} \int_{\tau_U} v, \quad \forall \tau_U \in T_U^h,\ v \in L^2(\Omega).$$

Similarly, (3.2) may be rewritten as

$$(\text{OCP–OPT}')^h \begin{cases} \text{(a) } a(q_h, p_h) = (f, q_h), & \forall q_h \in V^h, \\ \text{(b) } u_h = \max\{0, \mathcal{P}_h p_h\}, \\ \text{(c) } a(y_h, w_h) = (u_h, w_h), & \forall w_h \in V^h, \end{cases} \tag{3.4}$$

which are two the standard finite element systems.

In next sections, we will discuss the a priori error estimates and a posteriori estimator of the finite element approximation (OCT–OPT)^h.

4. A priori error estimate

We analyze a priori error estimates of the finite element approximation (OCT–OPT)^h. Let (y, p, u) and (y_h, p_h, p_h) be the solutions of (OCP–OPT) and (OCP–OPT)^h, respectively. We have the following error estimates.

Theorem 4.1. *There holds the a priori error estimate*

$$\|p - p_h\|_{L^2(\Omega)} + h_\tau \|\nabla(p - p_h)\|_{L^2(\Omega)} \leq C h_\tau^2 \|p\|_{H^2(\Omega)} \leq C h_\tau^2 \|f\|_{L^2(\Omega)}. \tag{4.1}$$

Proof. Noting that

$$a(p - p_h, q_h) = 0, \quad \forall q_h \in V^h$$

and by using the standard finite element methods, we have the error estimate (4.1). □

Theorem 4.2. *There holds the a priori error estimate*

$$\|u - u_h\|_{L^2(\Omega)} \leq C\left\{h_{\tau_U} + h_\tau^2\right\}. \tag{4.2}$$

Proof. Since

$$|u - u_h| = |\max\{0, p\} - \max\{0, \mathcal{P}_h p_h\}| \leq |p - \mathcal{P}_h p_h|,$$

then it follows from the boundedness of L^2-projection (see in [17]) that

$$\|u - u_h\|_{L^2(\Omega)} \leq \|p - \mathcal{P}_h p\|_{L^2(\Omega)} + \|\mathcal{P}_h(p - p_h)\|_{L^2(\Omega)} \leq C\left\{h_{\tau_U} \|p\|_{H^1(\Omega)} + h_\tau^2 \|p\|_{H^2(\Omega)}\right\}.$$

This ends the proof of Theorem 4.2. □

Lemma 4.1. *There holds the a priori error estimate*

$$\|y - y_h\|_{H^1(\Omega)} \leq C\left\{h_{\tau_U} + h_\tau\right\} \tag{4.3}$$

$$\|y - y_h\|_{L^2(\Omega)} \leq C\left\{h_{\tau_U} + h_\tau^2\right\}. \tag{4.4}$$

Proof. Let $\tilde{u} = \max\{0, \mathcal{P}_h p\}$. Introduce the auxiliary problem: $y_h(\tilde{u}) \in V^h$ such that

$$a(y_h(\tilde{u}), w_h) = (\tilde{u}, w_h), \quad \forall w_h \in V^h.$$

Noting that

$$a(y_h(\tilde{u}) - y_h, w_h) = (\tilde{u} - u_h, w_h), \quad \forall w_h \in V^h,$$

we have

$$\|\nabla(y_h(\tilde{u}) - y_h)\|_{L^2(\Omega)} \le C\|\tilde{u} - u_h\|_{L^2(\Omega)}.$$

It is clear that

$$\begin{aligned}\|\tilde{u} - u_h\|_{L^2(\Omega)} &= \|\max\{0, \mathcal{P}_h p\} - \max\{0, \mathcal{P}_h p_h\}\|_{L^2(\Omega)} \\ &\le \|\mathcal{P}_h(p - p_h)\|_{L^2(\Omega)} \le \|p - p_h\|_{L^2(\Omega)} \le Ch_\tau^2\|p\|_{H^2(\Omega)}.\end{aligned}$$

Thus we get

$$\|\nabla(y_h(\tilde{u}) - y_h)\|_{L^2(\Omega)} \le Ch_\tau^2\|p\|_{H^2(\Omega)}. \tag{4.5}$$

Let $\tilde{y}$ be defined by

$$-\Delta\tilde{y} = \tilde{u} \quad \text{in } \Omega,\ \tilde{y} = 0 \text{ on } \partial\Omega. \tag{4.6}$$

Since $y_h(\tilde{u})$ is the solution of the standard finite element scheme of the boundary value problem (4.6), hence we have the a priori error estimate

$$\|\tilde{y} - y_h(\tilde{u})\|_{L^2(\Omega)} + h_\tau\|\nabla(\tilde{y} - y_h(\tilde{u}))\|_{L^2(\Omega)} \le Ch_\tau^2\|\tilde{y}\|_{H^2(\Omega)} \le Ch_\tau^2\|p\|_{L^2(\Omega)}. \tag{4.7}$$

On the other hand, we have

$$a(\tilde{y} - y, w) = (\tilde{u} - u, w), \quad \forall w \in H_0^1(\Omega)$$

such that

$$\|\nabla(\tilde{y} - y)\|_{L^2(\Omega)}^2 = (\tilde{u} - u, \tilde{y} - y).$$

Let $\Omega_U^+ = \bigcup\{\tau_U \in T_U^h;\ p \ge 0\}$, and $\Omega_U^b = \Omega \setminus \Omega_U^+$. Then

$$(\tilde{u} - u, \tilde{y} - y) = (\mathcal{P}_h p - p, \tilde{y} - y)_{\Omega_U^+} + (\tilde{u} - u, \tilde{y} - y)_{\Omega_U^b}.$$

Noting that

$$|(\mathcal{P}_h p - p, \tilde{y} - y)_{\Omega_U^+}| = |(\mathcal{P}_h p - p, \tilde{y} - y - \mathcal{P}_h(\tilde{y} - y))_{\Omega_U^+}| \le Ch_{\tau_U}^2\|p\|_{H^1(\Omega)}\|\nabla(\tilde{y} - y)\|_{L^2(\Omega)}$$

and

$$|(\mathcal{P}_h p - p, \tilde{y} - y)_{\Omega_U^b}| \le Ch_{\tau_U}\|p\|_{H^1(\Omega_U^b)}\|\tilde{y} - y\|_{L^2(\Omega_U^b)}$$

we have

$$\|\nabla(\tilde{y} - y)\|_{L^2(\Omega)} \le Ch_{\tau_U}. \tag{4.8}$$

Noting that

$$\begin{aligned}\|y - y_h\|_{H^1(\Omega)} &\le C(\|y - \tilde{y}\|_{H^1(\Omega)} + \|\tilde{y} - y_h(\tilde{u})\|_{H^1(\Omega)} + \|y_h(\tilde{u}) - y_h\|_{H^1(\Omega)}) \\ \|y - y_h\|_{L^2(\Omega)} &\le C(\|y - \tilde{y}\|_{L^2(\Omega)} + \|\tilde{y} - y_h(\tilde{u})\|_{L^2(\Omega)} + \|y_h(\tilde{u}) - y_h\|_{L^2(\Omega)})\end{aligned}$$

and combined (4.5), (4.7) and (4.8), we proved (4.4). □

Furthermore, if we decompose Ω into three parts: $\Omega_1 = \{\tau_U : u|_{\tau_U} > 0\}$, $\Omega_2 = \{\tau_U : u|_{\tau_U} = 0\}$, $\Omega_3 = \Omega \setminus (\Omega_1 \cup \Omega_2)$, there exists the following superconvergence result (see [18], for details):

With the hypothesis that $\text{meas}(\Omega_3) \le Ch_{\tau_U}$, we have the optimal error estimate in L^2 norm:

$$\|y - y_h\|_{L^2(\Omega)} \le C\left\{h_{\tau_U}^2 + h_\tau^2\right\}. \tag{4.9}$$

5. Equivalent a posteriori error estimators

In this section we derive the a posterior error estimators equivalent to the error between the solution of the discrete problem and the solution of the infinite dimensional problem. We can see that y and p in the optimal control problem are smooth enough so that we do not need to use the adaptive finite element method to compute y, but sometimes it is useful to compute the curvature term Δy, which in our optimal control problem above is u indeed, and then it is necessary to introduce the adaptive finite element method to our analysis and computation. The numerical experiment in the next section may give us a good explanation.

The following lemmas are important in deriving a posteriori error estimates of residual type.

Lemma 5.1. *Let $\hat{\pi}_h$ be the average interpolation operator defined in* [19]. *For $m = 0$ or 1, $1 \le q \le \infty$ and $\forall v \in W^{1,q}(\Omega)$,*

$$|v - \hat{\pi}_h v|_{W^{m,q}(\tau)} \le \sum_{\bar{\tau}' \cap \bar{\tau} \neq \emptyset} C h_\tau^{1-m} |v|_{W^{1,q}(\tau')}, \quad \forall \tau \in T^h. \tag{5.1}$$

Lemma 5.2 (*[20]*)**.** *For each $v \in W^{1,q}(\Omega)$, $1 \le q < \infty$,*

$$\|v\|_{W^{0,q}(\partial\tau)} \le C \left(h_\tau^{-\frac{1}{q}} \|v\|_{W^{0,q}(\tau)} + h_\tau^{1-\frac{1}{q}} |v|_{W^{1,q}(\tau)} \right). \quad \forall \tau \in T^h \tag{5.2}$$

Lemma 5.3 (*[21]*)**.** *Let π_h be the standard Lagrange interpolation operator. For $m = 0$ or 1, $q > \frac{n}{2}$ and $v \in W^{2,q}(\Omega)$,*

$$|v - \pi_h v|_{W^{m,q}(\Omega^h)} \le C h^{2-m} |v|_{W^{2,q}(\Omega^h)}. \tag{5.3}$$

In order to derive sharp a posteriori error estimates, we divide Ω into some subsets:

$\Omega_0^+ = \{x \in \Omega : p_h(x) < 0, u_h > 0\}$,
$\Omega_0 = \{x \in \Omega : p_h(x) < 0, u_h = 0\}$,
$\Omega_0^- = \{x \in \Omega : p_h(x) \ge 0\}$.

Then, it is easy to see that the above three subsets are not intersected each other, and

$\bar{\Omega} = \bar{\Omega}_0^+ \cup \bar{\Omega}_0 \cup \bar{\Omega}_0^-$.

5.1. Equivalent a posteriori error estimators with H^1 norm

In the first part of this section we give the equivalent a posteriori error estimates in H^1 norms. Of course, here "in H^1 norms" actually means the estimators of $\|y - y_h\|_{H^1(\Omega)}$ and $\|p - p_h\|_{H^1(\Omega)}$.

Lemma 5.4. *There holds the estimate*

$$\|u - u_h\|_{L^2(\Omega)}^2 \le C(\eta_1^2 + \|p_h - p\|_{L^2(\Omega)}^2), \tag{5.4}$$

where

$$\eta_1^2 = \int_{\Omega_0^- \cup \Omega_0^+} (u_h - p_h)^2.$$

Proof. It is clear that

$$\begin{aligned} \|u - u_h\|_{L^2(\Omega)}^2 &= (u - p, u - u_h) - (u_h - p, u - u_h) \\ &\le (u_h - p, u_h - u) \\ &= (u_h - p - u_h + p_h, u_h - u) + (u_h - p_h, u_h - u) \\ &\le \frac{1}{2}\|p_h - p\|_{L^2(\Omega)}^2 + \frac{1}{2}\|u_h - u\|_{L^2(\Omega)}^2 + (u_h - p_h, u_h - u), \end{aligned} \tag{5.5}$$

where we used $(u - p, u - u_h) \le 0$, and we have

$$\|u - u_h\|_{L^2(\Omega)}^2 \le \|p_h - p\|_{L^2(\Omega)}^2 + 2(u_h - p_h, u_h - u). \tag{5.6}$$

It follows from the definition of Ω_0 that

$$\int_{\Omega_0} (u_h - p_h)(u_h - u) \le 0.$$

Then we have

$$\begin{aligned} \|u - u_h\|_{L^2(\Omega)}^2 &\le \|p_h - p\|_{L^2(\Omega)}^2 + 2\int_{\Omega_0^- \cup \Omega_0^+} (u_h - p_h)(u_h - u) \\ &\le \|p_h - p\|_{L^2(\Omega)}^2 + 2\int_{\Omega_0^- \cup \Omega_0^+} (u_h - p_h)^2 + \frac{1}{2}\|u - u_h\|_{L^2(\Omega)}^2. \end{aligned}$$

Hence,

$$\|u-u_h\|^2_{L^2(\Omega)} \le C\big\{\|p_h-p\|^2_{L^2(\Omega)}+\eta_1^2\big\}. \tag{5.7}$$

This ends the proof of Lemma 5.4. □

Theorem 5.1. *Let* (y,p,u) *and* (y_h,p_h,u_h) *be the solutions of* (OCP–OPT) *and* (OCP–OPT)^h, *Then there holds the estimate*

$$\|u-u_h\|^2_{L^2(\Omega)}+\|y-y_h\|^2_{H^1(\Omega)}+\|p-p_h\|^2_{H^1(\Omega)} \le C\sum_{i=1}^{3}\eta_i^2, \tag{5.8}$$

where

$$\eta_2^2=\sum_{\tau\in T^h}\int_\tau h_\tau^2 f^2+\sum_{l\cap\partial\Omega=\emptyset}\int_l h_l[\nabla p_h\cdot n]^2,$$
$$\eta_3^2=\sum_{\tau\in T^h}\int_\tau h_\tau^2 u_h^2+\sum_{l\cap\partial\Omega=\emptyset}\int_l h_l[\nabla y_h\cdot n]^2,$$

where l *is a face of an element* τ, n *is the unit normal vector on* $l=\bar\tau_l^1\cap\bar\tau_l^2$ *outwards* τ_l^1, h_l *is the maximum diameter of the face* l, *and* $[\nabla p_h\cdot n]$ *and* $[\nabla y_h\cdot n]$ *are the normal derivative jumps over the interior face* l, *defined by*

$$[\nabla p_h\cdot n]_l=(\nabla p_h|_{\tau_l^1}-\nabla p_h|_{\tau_l^2})\cdot n,$$
$$[\nabla y_h\cdot n]_l=(\nabla y_h|_{\tau_l^1}-\nabla y_h|_{\tau_l^2})\cdot n.$$

Proof. It follows from Lemma 5.4 that we need to estimate $\|p_h-p\|^2_{L^2(\Omega)}$. With the standard analysis of deriving a posteriori error estimator for a Poisson equation, such as in Chapter 1 in [22], we can easily obtain

$$\|p_h-p\|^2_{L^2(\Omega)} \le \|p-p_h\|^2_{H^1(\Omega)} \le C\eta_2^2.$$

It follows from Lemma 5.4 that

$$\|u-u_h\|^2_{L^2(\Omega)} \le C\sum_{i=1}^{2}\eta_i^2.$$

Similarly, we can prove

$$\|y(u_h)-y_h\|^2_{H^1(\Omega)} \le C\eta_3^2, \tag{5.9}$$

where $y(u_h)$ is the solution of

$$-\Delta y(u_h)=u_h,\quad y|_{\partial\Omega}=0. \tag{5.10}$$

Note that

$$\|y_h-y\|_{H^1(\Omega)} \le \|y_h-y(u_h)\|_{H^1(\Omega)}+\|y(u_h)-y\|_{H^1(\Omega)}, \tag{5.11}$$

and

$$\|y-y(u_h)\|^2_{H^1(\Omega)} \le C\|u-u_h\|^2_{L^2(\Omega)}. \tag{5.12}$$

Then

$$\|u-u_h\|^2_{L^2(\Omega)}+\|y-y_h\|^2_{H^1(\Omega)}+\|p-p_h\|^2_{H^1(\Omega)} \le C\sum_{i=1}^{3}\eta_i^2.$$

Then we proved Theorem 5.1. □

Using the standard bubble function technique (see [22,23], for example), and with the similar proof in [24], we derive the a posteriori lower bound and then the following equivalent a posteriori error estimates.

Theorem 5.2. *Let* (y,p,u) *and* (y_h,p_h,u_h) *be the solutions of* (OCP–OPT) *and* (OCP–OPT)^h, *then*

$$\eta_1^2+\eta_2^2+\eta_3^2 \le C(\|u-u_h\|^2_{L^2(\Omega)}+\|p-p_h\|^2_{H^1(\Omega)}+\|y-y_h\|^2_{H^1(\Omega)})+C\epsilon_1^2+Ce^2, \tag{5.13}$$

$$\|u-u_h\|^2_{L^2(\Omega)}+\|p-p_h\|^2_{H^1(\Omega)}+\|y-y_h\|^2_{H^1(\Omega)}+e^2 \le C(\eta_1^2+\eta_2^2+\eta_3^2) \tag{5.14}$$

where η_1 is defined in Lemma 5.4, η_2 and η_3 are defined in Theorem 5.1, and

$$e^2 = \int_{\Omega_*} ((u-p) - \mathcal{P}_h(u-p))^2, \qquad \epsilon_1^2 = \sum_{\tau\in T^h} \int_\tau h_\tau^2 (f - \bar{f})^2,$$

where $\mathcal{P}_h$ is the L^2-project operator from $L^2(\Omega)$ to U^h, $\Omega_* = \{x \in \Omega_0^+ : u(x) = 0,\ u_h(x) > 0\}$, and $\bar{v}|_\tau = \frac{\int_\tau v}{|\tau|}$.

Proof. The error estimates (5.13) for η_1 can be derived from Lemma 5.6 below, and the proof for η_2 and η_3 is similar. So we just give the proof on estimating the contributions of η_3 here.

Using the standard bubble function technique (see [22]), it can be proved that there exist polynomials $w_\tau \in H_0^1(\tau)$ and $w_l \in H_0^1(\tau_l^1 \cup \tau_l^2)$ such that

$$\int_\tau h_\tau^2 u_h^2 = \int_\tau u_h w_\tau, \tag{5.15}$$

$$\int_l h_l[(\nabla y_h \cdot n)]^2 = \int_l [(\nabla y_h \cdot n)] w_l, \tag{5.16}$$

and

$$\|w_\tau\|_{H^1(\tau)}^2 \le C \int_\tau h_\tau^2 u_h^2, \tag{5.17}$$

$$h_\tau^{-2} \|w_\tau\|_{L^2(\tau)}^2 \le C \int_\tau h_\tau^2 u_h^2, \tag{5.18}$$

$$\|w_l\|_{H^1(\tau_l^1 \cup \tau_l^2)}^2 \le C \int_l h_l[(\nabla y_h \cdot n)]^2, \tag{5.19}$$

$$h_l^{-2} \|w_l\|_{L^2(\tau_l^1 \cup \tau_l^2)}^2 \le C \int_l h_l[(\nabla y_h \cdot n)]^2. \tag{5.20}$$

Then it follows from (5.15), (5.17) and (5.18) that

$$\begin{aligned}
\int_\tau h_\tau^2 u_h^2 &= \int_\tau u_h \omega_\tau \\
&= \int_\tau u\omega_\tau + \int_\tau (u_h - u)\omega_\tau \\
&= \int_\tau (u - \Delta y - u)\omega_\tau + \int_\tau (u_h - u)\omega_\tau \\
&= \int_\tau (\Delta y_h - \Delta y)\omega_\tau + \int_\tau (u_h - u)\omega_\tau \\
&= \int_\tau \nabla(y - y_h)\nabla\omega_\tau + \int_\tau (u_h - u)\omega_\tau \\
&\le C(\delta)(\|y - y_h\|_{H^1(\tau)}^2 + \|u - u_h\|_{L^2(\tau)}^2) + C\delta(\|\omega_\tau\|_{H^1(\tau)}^2 + \|\omega_\tau\|_{L^2(\tau)}^2) \\
&\le C(\delta)(\|y - y_h\|_{H^1(\tau)}^2 + \|u - u_h\|_{L^2(\tau)}^2) + C_1\delta \int_\tau h_\tau^2 u_h^2
\end{aligned}$$

where C_1 is a general positive constant independent of h, and $\delta = \frac{1}{2C_1}$ is a small number. In the above proof, we used the property that y_h is a linear function in τ. Then we have

$$\sum_{\tau\in T^h} \int_\tau h_\tau^2 u_h^2 \le \sum_{\tau\in T^h} C(\|y - y_h\|_{H^1(\tau)}^2 + \|u - u_h\|_{L^2(\tau)}^2). \tag{5.21}$$

Similarly, when $l \cap \partial\Omega \ne \emptyset$, where $l = \bar{\tau}_1 \cap \bar{\tau}_2$, it follows from (5.16), (5.19) and (5.20) that

$$\begin{aligned}
h_l \int_l [(\nabla y_h \cdot n)]^2 &= \int_l [(\nabla y_h \cdot n)] w_l = \int_l [(\nabla y_h \cdot n) - (\nabla y \cdot n)] w_l \\
&= \int_{\partial\tau_l^1 \cup \partial\tau_l^2} ((\nabla y_h \cdot n) - (\nabla y \cdot n)) w_l
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\tau_l^1\cup\tau_l^2} \nabla(y_h - y)\nabla w_l + \int_{\tau_l^1\cup\tau_l^2} \Delta(y_h - y) w_l \\
&= \int_{\tau_l^1\cup\tau_l^2} \nabla(y_h - y)\nabla w_l + \int_{\tau_l^1\cup\tau_l^2} (\Delta y_h + u - u_h) w_l + \int_{\tau_l^1\cup\tau_l^2} u_h w_l \\
&\le C(\delta)\left(\|y - y_h\|^2_{H^1(\tau_l^1\cup\tau_l^2)} + \|u - u_h\|^2_{L^2(\tau_l^1\cup\tau_l^2)} + h_l^2\|u_h\|^2_{L^2(\tau_l^1\cup\tau_l^2)}\right) \\
&\quad + C\delta\left(\|\omega_l\|^2_{H^1(\tau_l^1\cup\tau_l^2)} + h_l^{-2}\|\omega_l\|^2_{L^2(\tau_l^1\cup\tau_l^2)}\right) \\
&\le C(\delta)\left(\|y - y_h\|^2_{H^1(\tau_l^1\cup\tau_l^2)} + \|u - u_h\|^2_{L^2(\tau_l^1\cup\tau_l^2)} + h_l^2\|u_h\|^2_{L^2(\tau_l^1\cup\tau_l^2)}\right) + C\delta\int_l h_l[(\nabla y_h\cdot n)]^2,
\end{aligned}
$$

which deduced that

$$
h_l\int_l [(\nabla y_h\cdot n)]^2 \le C(\delta)\left(\|y - y_h\|^2_{H^1(\tau_l^1\cup\tau_l^2)} + \|u - u_h\|^2_{L^2(\tau_l^1\cup\tau_l^2)} + h_l^2\|u_h\|^2_{L^2(\tau_l^1\cup\tau_l^2)}\right). \tag{5.22}
$$

Combined with (5.21) and (5.22), we have

$$
\eta_3^2 = \sum_{\tau\in T^h}\int_\tau h_\tau^2 u_h^2 + \sum_{l\cap\partial\Omega\neq\emptyset} h_l\int_l [(\nabla y_h\cdot n)]^2 \le C(\delta)\left(\|y - y_h\|^2_{H^1(\Omega)} + \|u - u_h\|^2_{L^2(\Omega)}\right).
$$

Then we proved (5.13), and Theorem 5.2 follows from Theorem 5.1. □

5.2. Equivalent a posteriori error estimators with L^2 norm

In the rest of this section we give our equivalent a posteriori error estimates in L^2 norms. First, we need an a priori regular estimate for the following auxiliary problem:

$$
-\Delta\xi = f_1 \quad \text{in } \Omega,\ \xi|_{\partial\Omega} = 0, \tag{5.23}
$$

$$
-\Delta\zeta = f_2 \quad \text{in } \Omega,\ \zeta|_{\partial\Omega} = 0. \tag{5.24}
$$

Lemma 5.5 ([25]). *Let ξ and ζ be the solutions of (5.23) and (5.24) respectively. Assume that Ω is convex, then*

$$
\|\xi\|_{H^2(\Omega)} \le C\|f_1\|_{L^2(\Omega)}, \qquad \|\zeta\|_{H^2(\Omega)} \le C\|f_2\|_{L^2(\Omega)}.
$$

Theorem 5.3. *Let (y, p, u) and (y_h, p_h, u_h) be solutions of* (OCP–OPT) *and* (OCP–OPT)^h*, respectively. Then*

$$
\|u - u_h\|^2_{L^2(\Omega)} + \|y - y_h\|^2_{L^2(\Omega)} + \|p - p_h\|^2_{L^2(\Omega)} \le C(\eta_1^2 + \hat{\eta_2}^2 + \hat{\eta_3}^2), \tag{5.25}
$$

where η_1 is defined in Lemma 5.4 and

$$
\hat{\eta_2}^2 = \sum_{\tau\in T^h} h_\tau^4\int_\tau f^2 + \sum_{l\cap\partial\Omega=\emptyset} h_l^3\int_l [\nabla p_h\cdot n]^2,
$$

$$
\hat{\eta_3}^2 = \sum_{\tau\in T^h} h_\tau^4\int_\tau u_h{}^2 + \sum_{l\cap\partial\Omega=\emptyset} h_l^3\int_l [\nabla y_h\cdot n]^2.
$$

Proof. We only need to estimate the error for y and p. Let ζ be the solution of (5.24) with $f_2 = p - p_h$, let $\zeta_I = \pi_h\zeta \in V^h$ be the standard Lagrange interpolation of ζ. Then it follows from Lemmas 5.1, 5.5, 5.3 and also Young's inequality that

$$
\begin{aligned}
\|p - p_h\|^2_{L^2(\Omega)} &= (\nabla\zeta, \nabla(p - p_h)) \\
&= (\nabla(\zeta - \zeta_I), \nabla(p - p_h)) + (\nabla\zeta_I, \nabla(p - p_h)) \\
&= \sum_{\tau\in T^h}\int_\tau (f + \Delta p_h)(\zeta - \zeta_I) - \sum_{l\cap\partial\Omega=\emptyset}\int_l [\nabla p_h\cdot n](\zeta - \zeta_I)ds \\
&\le C\sum_{\tau\in T^h} h_\tau^2\|f\|_{L^2(\tau)}|\zeta|_{H^2(\tau)} + C\sum_{l\cap\partial\Omega=\emptyset} h_l^{\frac{3}{2}}\left(\int_l [\nabla p_h\cdot n]^2\right)^{\frac{1}{2}}\|\zeta\|_{H^2(\tau_l^1\cup\tau_l^2)} \\
&\le C(\delta)\sum_{\tau\in T^h} h_\tau^4\int_\tau f^2 + C(\delta)\sum_{l\cap\partial\Omega=\emptyset} h_l^3\int_l [\nabla p_h\cdot n]^2 + C_2\delta\|p - p_h\|^2_{L^2(\Omega)}
\end{aligned}
$$

where C_2 is a general positive constant independent of h, and $\delta = \frac{1}{2C_2}$ is a small number. Then we have

$$\|p - p_h\|^2_{L^2(\Omega)} \le C \sum_{\tau \in T^h} h_\tau^4 \int_\tau f^2 + C \sum_{l \cap \partial\Omega = \emptyset} h_l^3 \int_l [\nabla p_h \cdot n]^2. \tag{5.26}$$

Similarly, let ξ be the solution of (5.23) with $f_1 = y(u_h) - y_h$, where $y(u_h)$ is the solution of (5.10). Then

$$\begin{aligned}\|y(u_h) - y_h\|^2_{L^2(\Omega)} &= (\nabla(y(u_h) - y_h), \nabla\xi) \\ &\le C(\delta) \sum_{\tau \in T^h} h_\tau^4 \int_\tau u_h^2 + C(\delta) \sum_{l \cap \partial\Omega = \emptyset} h_l^3 \int_l [\nabla y_h \cdot n]^2 + C\delta \|y(u_h) - y_h\|^2_{L^2(\Omega)}.\end{aligned}$$

Hence,

$$\|y(u_h) - y_h\|^2_{L^2(\Omega)} \le C \sum_{\tau \in T^h} h_\tau^4 \int_\tau (u_h + \Delta y_h)^2 + C \sum_{l \cap \partial\Omega = \emptyset} h_l^3 \int_l [\nabla y_h \cdot n]^2. \tag{5.27}$$

Note that

$$\begin{aligned}\|y_h - y\|_{H^1(\Omega)} &\le \|y_h - y(u_h)\|_{H^1(\Omega)} + \|y(u_h) - y\|_{H^1(\Omega)} \\ &\le \|y_h - y(u_h)\|_{H^1(\Omega)} + \|u - u_h\|_{L^2(\Omega)},\end{aligned} \tag{5.28}$$

then (5.25) follows from (5.26), (5.27), (5.28) and Theorem 5.1. □

Now we are in the position of deriving the a posteriori lower bounds with L^2 norm.

Lemma 5.6. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of* (OCP–OPT) *and* (OCP–OPT)^h, *then*

$$\eta_1^2 \le C(e^2 + \|u - u_h\|^2_{L^2(\Omega)} + \|p - p_h\|^2_{L^2(\Omega)})$$

where e^2 is defined in Theorem 5.2.

Proof. To make it clear and complete, we repeat the similar proof in [24]. Let $\Omega_0^d = \{x \in \Omega_0^- : u(x) = 0\}$, and note that

$$\begin{cases} u - p = 0, & \text{when } u > 0; \\ u - p \ge 0, & \text{when } u = 0. \end{cases}$$

Then we can derive that

$$\begin{aligned}\int_{\Omega_0^-} (u_h - p_h)^2 &= \int_{\Omega_0^d} (u_h - p_h)^2 + \int_{\Omega_0^- \setminus \Omega_0^d} (u_h - p_h)^2 \\ &= \int_{\Omega_0^d} (u_h - p_h - u)^2 + \int_{\Omega_0^- \setminus \Omega_0^d} (u_h - p_h - u + p)^2 \\ &\le C \left(\|u - u_h\|^2_{L^2(\Omega)} + \|p - p_h\|^2_{L^2(\Omega)} + \int_{\Omega_0^d} {p_h}^2 \right) \\ &\le C \left(\|u - u_h\|^2_{L^2(\Omega)} + \|p - p_h\|^2_{L^2(\Omega)} + \int_{\Omega_0^d} (p_h - p)^2 \right) \\ &\le C(\|u - u_h\|^2_{L^2(\Omega)} + \|p - p_h\|^2_{L^2(\Omega)}),\end{aligned}$$

where we used $p_h \ge 0$ and $p \le 0$ on Ω_0^d in the second last step.

Recall that

$$\Omega_* = \{x \in \Omega_0^+ : u(x) = 0,\ u_h(x) > 0\},$$

and note $u > 0$, on $\Omega_0^+ \setminus \Omega_*$, so $u - p = 0$, on $\Omega_0^+ \setminus \Omega_*$. Since $u_h > 0$ on Ω_*, then it follows from (3.4)(b) that $u_h = \max(0, \mathcal{P}_h p_h) = \mathcal{P}_h p_h > 0$ on Ω_*, so $\mathcal{P}_h(u_h - p_h) = u_h - \mathcal{P}_h p_h = 0$ on Ω_*.

$$\begin{aligned}\int_{\Omega_0^+} (u_h - p_h)^2 &= \int_{\Omega_*} (u_h - p_h)^2 + \int_{\Omega_0^+ \setminus \Omega_*} (u_h - p_h)^2 \\ &= \int_{\Omega_*} ((u_h - p_h) - \mathcal{P}_h(u_h - p_h))^2 + \int_{\Omega_0^+ \setminus \Omega_*} (u_h - p_h - u + p)^2\end{aligned}$$

$$
\begin{aligned}
&\le C\int_{\Omega_*}((u_h-p_h)-(u-p))^2+C\int_{\Omega_*}((u-p)-\mathcal{P}_h(u-p))^2\\
&\quad+C\int_{\Omega_*}(\mathcal{P}_h(u-p)-\mathcal{P}_h(u_h-p_h))^2+C(\|p-p_h\|^2_{L^2(\Omega)}+\|u-u_h\|^2_{L^2(\Omega_U)})\\
&\le Ce^2+C(\|p-p_h\|^2_{L^2(\Omega)}+\|u-u_h\|^2_{L^2(\Omega_U)}).
\end{aligned}
$$

Then Lemma 5.6 is proved. □

The following results are based on a modification (introduced in [26]) of the standard bubble function technique (see [22,23], for example). First, we introduce some notations.

Denote by $\lambda_{\tau,1},\dots,\lambda_{\tau,3}$ the barycentric co-odinates of $\tau\in T^h$. We then define the triangle-bubble function b_τ by

$$
b_\tau=\begin{cases}\lambda_{\tau,1}\lambda_{\tau,2}\lambda_{\tau,3}, & \text{on } \tau,\\ 0, & \text{on } \Omega\setminus\tau.\end{cases}\tag{5.29}
$$

Given an edge $l=\tau_1\cap\tau_2$, where $\tau_1,\tau_2\in T^h$, we enumerate the vertices of τ_1 and τ_2 such that the vertices of l are numbered first. We then define the edge-bubble function b_l by

$$
b_l=\begin{cases}\lambda_{\tau_1,1}\lambda_{\tau_1,2}\lambda_{\tau_2,1}\lambda_{\tau_2,2}, & \text{on } \tau_1\cup\tau_2,\\ 0, & \text{on } \Omega\setminus(\tau_1\cup\tau_2).\end{cases}\tag{5.30}
$$

Here we give some properties of the bubble-functions b_τ and b_l, which were deduced from [22,26]. Its proof immediately follows from Definitions (5.29) and (5.30) and the standard scaling arguments for finite element functions. So we just give the following Lemma 5.7 without proof.

Lemma 5.7. *Let b_τ and b_l be the bubble-functions defined in (5.29) and (5.30), then*

$$
\begin{aligned}
&0\le|b_\tau|\le 1,\quad 0\le|b_l|\le 1;\\
&\operatorname{supp} b_\tau\subseteq\tau,\qquad \operatorname{supp} b_l\subseteq\tau_1\cup\tau_2,\ b_\tau\in H_0^2(\tau),\quad b_l\in H_0^2(\tau_1\cup\tau_2);\\
&c_1h_\tau^{-2}\|b_\tau^2\|_{L^2(\tau)}\le\|b_\tau^2\|_{H^2(\tau)}\le c_2h_\tau^{-2}\|b_\tau^2\|_{L^2(\tau)};\\
&c_3h_l^{-2}\|b_l^2\|_{L^2(\tau_1\cup\tau_2)}\le\|b_l^2\|_{H^2(\tau_1\cup\tau_2)}\le c_4h_l^{-2}\|b_l^2\|_{L^2(\tau_1\cup\tau_2)}.
\end{aligned}
$$

Lemma 5.8. *Let (y,p,u) and (y_h,p_h,u_h) be the solutions of* (OCP–OPT) *and* (OCP–OPT)^h, *then*

$$
\sum_{\tau\in T^h}h_\tau^4f^2\le C(\|p-p_h\|^2_{L^2(\Omega)}+\epsilon_2^2),
$$

where

$$
\epsilon_2^2=\sum_{\tau\in T^h}\int_\tau h_\tau^4(f-\bar f)^2.
$$

Proof. Note that

$$
h_\tau^4\int_\tau\bar f^2=h_\tau^4\int_\tau\bar f\omega_\tau,
$$

where $\omega_\tau=c_5\bar f b_\tau^2$, and $c_5=\frac{\int_\tau 1}{\int_\tau b_\tau^2}$ is a constant. It follows from Lemma 5.7 that $\omega_\tau\in H_0^2(\tau)$. Then

$$
\begin{aligned}
h_\tau^4\int_\tau\bar f^2&=\int_\tau h_\tau^4\bar f\omega_\tau\\
&=\int_\tau h_\tau^4f\omega_\tau+\int_\tau h_\tau^4(\bar f-f)\omega_\tau\\
&\le\int_\tau h_\tau^4(f-\Delta p-f)\omega_\tau+C(\delta)\int_\tau h_\tau^4(f-\bar f)^2+C\delta h_\tau^4\|\omega_\tau\|^2_{L^2(\tau)}\\
&=\int_\tau h_\tau^4(\Delta p_h-\Delta p)\omega_\tau+C(\delta)\int_\tau h_\tau^4(f-\bar f)^2+C\delta h_\tau^4\|\omega_\tau\|^2_{L^2(\tau)}\\
&=\int_\tau h_\tau^4(\nabla p-\nabla p_h)\nabla\omega_\tau+C(\delta)\int_\tau h_\tau^4(f-\bar f)^2+Ch_\tau^4\delta\|\omega_\tau\|^2_{L^2(\tau)}
\end{aligned}
$$

$$
\begin{aligned}
&= \int_\tau h_\tau^4 (p_h - p)\Delta\omega_\tau + C(\delta)\int_\tau h_\tau^4 (f-\bar{f})^2 + C\delta h_\tau^4 \|\omega_\tau\|^2_{L^2(\tau)} \\
&\le C(\delta)\left(\|p-p_h\|^2_{L^2(\tau)} + \int_\tau h_\tau^4 (f-\bar{f})^2\right) + C\delta h_\tau^4 \|\omega_\tau\|^2_{L^2(\tau)} + C\delta h_\tau^8 \|\omega_\tau\|^2_{H^2(\tau)} \\
&\le C(\delta)\left(\|p-p_h\|^2_{L^2(\tau)} + \int_\tau h_\tau^4 (f-\bar{f})^2\right) + C\delta h_\tau^4 \bar{f}^2 \|b_\tau^2\|^2_{L^2(\tau)} + C\delta h_\tau^8 \bar{f}^2 \|b_\tau^2\|^2_{H^2(\tau)} \\
&\le C(\delta)\left(\|p-p_h\|^2_{L^2(\tau)} + \int_\tau h_\tau^4 (f-\bar{f})^2\right) + C\delta \int_\tau h_\tau^4 \bar{f}^2,
\end{aligned}
$$

where $0 < \delta \ll 1$, and we used Lemma 5.7 in the last step. So we have the estimate that

$$
\int_\tau h_\tau^4 \bar{f}^2 \le C\left(\|p-p_h\|^2_{L^2(\tau)} + \int_\tau h_\tau^4 (f-\bar{f})^2\right).
$$

Note that

$$
\int_\tau h_\tau^4 f^2 \le C\int_\tau h_\tau^4 \bar{f}^2 + C\int_\tau h_\tau^4 (f-\bar{f})^2,
$$

then Lemma 5.8 is proved. □

Lemma 5.9. *Let* (y, p, u) *and* (y_h, p_h, u_h) *be the solutions of* (OCP–OPT) *and* $(\text{OCP–OPT})^h$, *then*

$$
\sum_{l\cap\partial\Omega=\emptyset} h_l^3 \int_l [\nabla p_h \cdot n]^2 \le C(\|p-p_h\|^2_{L^2(\Omega)} + \epsilon_2^2).
$$

Proof. Note that

$$
h_l^3 \int_l [\nabla p_h \cdot n]^2 = h_l^3 \int_l [\nabla p_h \cdot n]\omega_l,
$$

where $\omega_l = c_6[\nabla p_h \cdot n]b_l^2$, and $c_6 = \frac{\int_\tau 1}{\int_\tau b_l^2}$ is a constant. It follows from Lemma 5.7 that $\omega_l \in H_0^2(\tau_1 \cup \tau_2)$. Then

$$
\begin{aligned}
h_l^3 \int_l [\nabla p_h \cdot n]^2 &= \int_l h_l^3 [\nabla p_h \cdot n] w_l = \int_l h_l^3 [(\nabla p_h \cdot n) - (\nabla p \cdot n)] w_l \\
&= \int_{\partial\tau_l^1 \cup \partial\tau_l^2} h_l^3 ((\nabla p_h \cdot n) - (\nabla p \cdot n)) w_l \\
&= \int_{\tau_l^1\cup\tau_l^2} h_l^3 \nabla(p_h - p)\nabla w_l + \int_{\tau_l^1\cup\tau_l^2} h_l^3 \Delta(p_h - p) w_l \\
&= \int_{\tau_l^1\cup\tau_l^2} h_l^3 (p - p_h)\Delta w_l + \int_{\tau_l^1\cup\tau_l^2} h_l^3 (\Delta p_h + f) w_l \\
&\le C(\delta)\|p_h - p\|^2_{L^2(\tau_l^1\cup\tau_l^2)} + C(\delta)\int_{\tau_l^1\cup\tau_l^2} h_l^4 f^2 + C\delta\left(h_l^6 \|w_l\|^2_{H^2(\tau_l^1\cup\tau_l^2)} + h_l^2 \|w_l\|^2_{L^2(\tau_l^1\cup\tau_l^2)}\right) \\
&\le C(\delta)\|p_h - p\|^2_{L^2(\tau_l^1\cup\tau_l^2)} + C(\delta)\int_{\tau_l^1\cup\tau_l^2} h_l^4 f^2 + C\delta\left(h_l^6 [\nabla p_h \cdot n]^2 \|b_l^2\|^2_{H^2(\tau_l^1\cup\tau_l^2)}\right. \\
&\quad \left. + h_l^2 [\nabla p_h \cdot n]^2 \|b_l^2\|^2_{L^2(\tau_l^1\cup\tau_l^2)}\right).
\end{aligned}
$$

From the standard scaling arguments, it is easy to see that

$$
\|b_l^2\|_{L^2(\tau_l^1\cup\tau_l^2)} \le C h_\tau^{\frac{1}{2}} \|b_l^2\|_{L^2(l)},
$$

where $l = \tau_l^1 \cap \tau_l^2$. Combined with Lemmas 5.7 and 5.8, we have the estimate that

$$
h_l^3 \int_l [\nabla p_h \cdot n]^2 \le C(\|p-p_h\|^2_{L^2(\tau_l^1\cup\tau_l^2)} + \epsilon_2^2).
$$

Then Lemma 5.9 is proved. □

Table 1
L^2 error of y on uniform mesh.

	Nodes	Sides	Elements	L^2 error of y
$h = 0.1$	140	377	238	8.153e−04
$h = 0.05$	507	1438	932	2.094e−04
$h = 0.025$	1930	5627	3698	5.264e−05

From Lemmas 5.8 and 5.9 we obtain the following conclusion.

Lemma 5.10. *Let* (y, p, u) *and* (y_h, p_h, u_h) *be the solutions of* (OCP–OPT) *and* (OCP–OPT)^h, *then*

$$\hat{\eta}_2^2 \le C(\|p - p_h\|^2_{L^2(\Omega)} + \epsilon_2^2). \tag{5.31}$$

Similarly, we can prove the following lower bound estimate.

Lemma 5.11. *Let* (y, p, u) *and* (y_h, p_h, u_h) *be the solutions of* (OCP–OPT) *and* (OCP–OPT)^h, *then*

$$\hat{\eta}_3^2 \le C(\|y - y_h\|^2_{L^2(\Omega)} + \|u - u_h\|^2_{L^2(\Omega)}). \tag{5.32}$$

As a corollary of Lemmas 5.10 and 5.11, we have the following theorem.

Theorem 5.4. *Let* (y, p, u) *and* (y_h, p_h, u_h) *be the solutions of* (OCP–OPT) *and* (OCP–OPT)^h, *respectively. Then*

$$\eta_1^2 + \hat{\eta}_2{}^2 + \hat{\eta}_3{}^2 \le C(\|u - u_h\|^2_{L^2(\Omega)} + \|y - y_h\|^2_{L^2(\Omega)} + \|p - p_h\|^2_{L^2(\Omega)} + e^2 + \epsilon_2^2), \tag{5.33}$$

where η_1^2, $\hat{\eta}_2{}^2$ *and* $\hat{\eta}_3{}^3$ *are defined in Theorem 5.3, and* ϵ_2 *is defined in Lemma 5.8, and* e^2 *is defined in Theorem 5.2.*

6. Numerical experiments

In this section, we carry out some numerical experiments to demonstrate the error estimates developed in Sections 4 and 5. In computing the solutions, we used the software package: AFEpack, see [27] for the details.

Example 1. In this example we consider the variational inequality problem (2.1):

$$\begin{cases} \text{Find } y \in K, \quad \text{such that} \\ \displaystyle\int_\Omega \Delta y \Delta(w - y) \ge \int_\Omega f(w - y), \quad \forall w \in K \end{cases}$$

where $K = \{y \in H^2(\Omega) \cap H_0^1(\Omega) : \Delta y \le 0 \text{ in } \Omega\}$.

$$\Omega = (0, 1)^2$$
$$f = 2\pi^2 \sin(\pi x_1) \sin(\pi x_2)$$
$$y = \frac{1}{2\pi^2} \sin(\pi x_1) \sin(\pi x_2).$$

The numerical results are in Table 1:

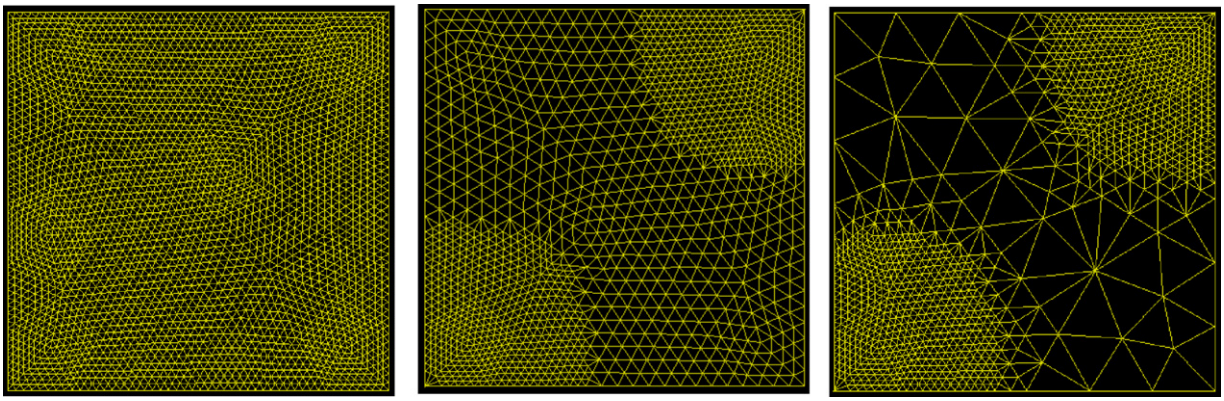
In the results showing in Table 1 we can see clearly that the error of y is of 2 order when $h_\tau = h_{\tau_U} = h$, which coincides with the superconvergence results in (4.9). This is because in this example we have $-\Delta y = u = p \ge 0$ in Ω, so that the hypothesis $\text{meas}(\Omega_3) \le \text{Ch}_U$ is satisfied. As far as the authors know, it is almost the first time a numerical result of the (2.1)-type fourth-order variational inequality with curvature obstacle has been given, even though the solution y is in a special form that $\Delta y < 0$. In this example, we first give the exact form of y, and it is not easy to construct such a solution with low regularity properties, so in the next example, we will first give the exact form of u which has low regularity.

Example 2. Sometimes it is useful to calculate the curvature Δy, so in this example we turn the variational inequality (2.1) into the following type of optimal control problem and calculate u via the adaptive finite element method:

$$\min_{u \in K' = \{u : u \ge 0\}} J(y, u) = \frac{1}{2} \int_\Omega u^2 - \int_\Omega fy$$
$$\text{s.t.} \begin{cases} -\Delta y = u, & \text{in } \Omega \\ y = 0, & \text{on } \partial\Omega. \end{cases}$$

Table 2
The error estimates at different refinement steps.

		Initial uniform mesh	Adaptive mesh step-1		Last adaptive mesh		*jy* mesh
		uyp mesh	*u* mesh	*yp* mesh	*u* mesh	*yp* mesh	
Mesh	# nodes	2913	1542	753	1171	753	180 993
	# edges	8544	4431	2160	3316	2160	541 440
	# elements	5632	2890	1408	2146	1408	359 424
Error	$\\|u-u_h\\|_{L^2(\Omega)}$	7.599e−3	7.863e−3		7.847e−3		
	$\\|y-y_h\\|_{L^2(\Omega)}$	1.976e−5	6.204e−5		6.204e−5		
	$\\|p-p_h\\|_{L^2(\Omega)}$	1.100e−3	4.386e−3		4.386e−3		
	$\\|y-y_h\\|_{H^1(\Omega)}$	1.147e−4	4.570e−4		4.569e−3		
	$\\|p-p_h\\|_{H^1(\Omega)}$	2.074e−1	4.142e−1		4.142e−1		



the initial mesh of u *the 1 − step adaptive mesh of u* *the last adaptive mesh of u*

Fig. 1. The meshes at different refinement steps.

and the co-state equation is $-\Delta p = f$, in Ω, $p|_{\partial\Omega} = 0$. We take

$$\begin{aligned}
&\Omega = (0, 1)^2\\
&p = \sin(2\pi x_1)\sin(2\pi x_2) - \sin(\pi x_1)\sin(\pi x_2)\\
&f = 8\pi^2\sin(2\pi x_1)\sin(2\pi x_2) - 2\pi^2\sin(\pi x_1)\sin(\pi x_2)\\
&u = \max(0, p).
\end{aligned}$$

We now briefly describe the solution algorithm to be used for solving this example:

Algorithm
(i) solve the discretized optimization problem with the projection gradient method on the current meshes and calculate the error estimators η_i or $\hat{\eta}_i$;
(ii) adjust the meshes using the estimators and update the solution on new meshes, as described. The details on how to adjust the meshes can be seen in [16,27], etc.

In this example, we use η_1 as the control mesh refinement indicator, and $\eta_2 + \eta_3$ as the state's and co-state's for H^1-*norm*, and $\hat{\eta}_2 + \hat{\eta}_3$ as the state's and co-state's for L^2-*norm*. In this example, it is very difficult for us to obtain the exact solution of y, so we cannot get the L^2 error for $y - y_h$. However, we can calculate the FEM solution of $(-\Delta y = u,\ u|_{\partial\Omega} = 0)$ on an deeply refined mesh to be the quasi-exact solution of y, which is denoted as jy in the following. We use the norm $\|y_h - jy\|_{L^2(\Omega)}$ to substitute for $\|y_h - y\|_{L^2(\Omega)}$ where y is the exact solution of $(-\Delta y = u,\ u|_{\partial\Omega} = 0)$. So in Table 2, the term $\|y_h - y\|_{L^2(\Omega)}$ is actually $\|y_h - jy\|_{L^2(\Omega)}$.

The error estimates at initial mesh, adaptive mesh step 1, and the last mesh step are as follows.

With the results shown in Table 2 we can see that we get almost the same accuracy in $\|u-u_h\|_{L^2(\Omega)}$ on adaptive mesh, with more than 60% of grids saved. As the control problem (OCP) has been translated to the control of u, but not the control of y, so we may obtain same or better results in u with adaptive finite elements method, and meanwhile save lots of computational work. Here we give the meshes of u at the refinement steps mentioned above, respectively.

From Fig. 1 we can see that with the indicators' affect, the grids are refined near the area where the control u has lack of regularity, and coarsened in other area. With the results in Table 2 and Fig. 1, the convergence of our scheme can be shown.

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