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Comparison of Estimation Methods for Frechet Distribution with Known Shape

Kamran Abbas[1,2,*], Tang Yincai[2]
[1]Department of Statistics, University of Azad Jammu and Kashmir, Muzaffarabad, Pakistan
[2]School of Finance and Statistics, East China Normal University, Shanghai, China
*Corresponding Author: Email: kamiuajk@gmail.com

This paper deals with the estimation of scale parameter for Frechet distribution with known shape. The Maximum likelihood estimation and Probability weighted moment estimation are discussed. Bayes estimator is obtained using Jeffreys' prior under quadratic loss function, El-Sayyad's loss function and linex loss function. Through extensive simulation study, we compared the performance of these estimators considering various sample size based on mean squared error (MSE).

Key Words: Maximum likelihood estimator, Probability weighted moment estimator, Mean squared error, Loss function, Frechet distribution.

1. INTRODUCTION

Frechet distribution was introduced by a French mathematician named Maurice Frechet (1878-1973) who had identified before one possible limit distribution for the largest order statistic in 1927. The Frechet distribution has been shown to be useful for modeling and analysis of several extreme events ranging from accelerated life testing to earthquakes, floods, rain fall, sea currents and wind speeds.

Applications of the Frechet distribution in various fields given in Harlow (2002) showed that it is an important distribution for modeling the statistical behavior of materials properties for a variety of engineering applications. Nadarajah and Kotz (2008) discussed the sociological models based on Frechet random variables. Further, Zaharim et al. (2009) applied Frechet distribution for analyzing the wind speed data. Mubarak (2011) studied the Frechet progressive type-II censored data with binomial removals.

The Frechet distribution is a special case of the generalized extreme value distribution. This type-II extreme value distribution (Frechet) case is equivalent to taking the reciprocal of values from a standard Weibull distribution. The probability density function (PDF) and the cumulative distribution function (CDF) for Frechet distribution is

$$f(x,\alpha,\beta)=\frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{\alpha+1}\exp\left[-\left(\frac{\beta}{x}\right)^{\alpha}\right],\quad x>0 \qquad (1)$$

where the parameter $\alpha>0$ determines the shape of the distribution and $\beta>0$ is the scale parameter.

$$F(x)=\exp\left[-\left(\frac{\beta}{x}\right)^{\alpha}\right],\quad x>0 \qquad (2)$$

Several methods have been proposed to estimate the parameters using both classical and Bayesian techniques. A method of estimation must be chosen which minimizes sampling errors. A method which is suitable to estimate the parameters of one distribution might not necessarily be as efficient for another distribution. Moreover, a method which is efficient in estimating the parameters may not be efficient in predicting given by Al-Baidhani and Sinclair (1987). Ahmed et al. (2010) have considered ML and Bayesian estimation of the scale parameter of Weibull distribution with known shape and compared their performance under squared error loss. In this paper, comparison among the ML estimator, probability weighted moment estimator (PWM) and Bayes estimator of the scale parameter of the Frechet distribution is considered under quadratic loss function, El-Sayyad's loss function and linex loss function using Jeffreys' prior, with the assumption that the shape parameter is known.

Comparisons are made in terms of the bias and mean squared error (MSE) of the estimates.

The plan of the paper is as follows. In Section 2, the ML estimation of β is reviewed. In Section 3 of this article, we derive the Bayes estimators based on squared error loss function, El-Sayyad's loss function and linex loss function. In Section 4, we obtain the probability weighted moment estimator and in Section 5, a simulation study is discussed. In Section 6, conclusion and numerical results are presented.

2. MAXIMUM LIKELIHOOD ESTIMATION

Let X = (x_1, x_2….x_n) be a sample of size n from a Frechet distribution with parameters α and β. The likelihood function is given by

$$L_n(\alpha,\beta) = \alpha^n \beta^{n\alpha} \prod_{i=1}^{n} x_i^{-(\alpha+1)} \exp\left[-\sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\alpha}\right]$$

As the shape parameter α is assumed to be known, the ML estimator of β is obtained by solving the following equation

$$\frac{\partial \log L_n(\alpha,\beta)}{\partial\beta} = 0,$$

Therefore, the maximum likelihood estimator for β is

$$\hat{\beta}_{ML} = \left[\frac{n}{t}\right]^{1/\alpha}, \quad where \ \ t = \sum_{i=1}^{n} (1/x_i)^{\alpha}$$

3. BAYESIAN ESTIMATION

For Bayesian estimation, we need prior distribution for scale parameter β. Jeffreys (1946) proposed a formal rule for obtaining a non-informative prior as

$$\Pi_J(\beta) \propto \sqrt{Det[I(\beta)]}$$

where β is M-vector valued parameter and I(β) is the Fisher's information matrix of order M × M. In the present study, we consider the prior distribution of β to be

$$\Pi_J(\beta) \propto \sqrt{Det[I(\beta)]} = \frac{k}{\beta}$$

where k is constant. The posterior distribution of β is

$$f(\beta \mid \underline{x}) \propto f(\underline{x} \mid \beta)\Pi_J(\beta)$$

Therefore, the posterior density function of β^{α} given α is Gamma $\left(n, \sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\alpha}\right)$

$$f(\beta \mid \underline{x}) = \frac{t^n}{\Gamma(n)}\beta^{n-1}e^{-t\beta}$$

3.1. Loss Functions

We consider three types of loss functions. The first is the squared loss function (quadratic loss). This loss function is symmetric in nature. i.e. it gives equal importance to both over and under estimation. The squared error loss function is as follows

$$L_1(\hat{\beta},\beta) = (\hat{\beta}-\beta)^2$$

The second is the El-Sayyad's loss function and is expressed as

$$L_2(\hat{\beta},\beta) = \beta^l(\hat{\beta}^r - \beta^r)^2$$

Where $\hat{\beta}$ is an estimator of β. It is remarkable that a number of asymmetric loss functions are proposed for use, among these, one of the most useful asymmetric loss known as the linex (linear exponential) loss function was introduced by Varian in (1975) given by

$$L_3(\hat{\beta}, \beta) = \exp[-c\Delta] - c\Delta - 1; \quad \Delta = \hat{\beta} - \beta, \ c \neq 0$$

Here, c determines the shape of the loss function. If $c > 0$, the overestimation is more serious than underestimation and vice-versa. For 'c' close to zero, the linex loss is approximately squared error loss.

3.2. Estimation under squared error loss

To obtain the Bayes estimator, we minimize the posterior expected loss given by

$$\begin{aligned} L_1 &= \int_0^\infty (\hat{\beta} - \beta)^2 \frac{t^n}{\Gamma(n)} \beta^{n-1} e^{-t\beta} d\beta \\ L_1 &= \hat{\beta}^2 + \frac{n(n+1)}{t^2} - \frac{2\hat{\beta}n}{t} \end{aligned}$$

Solving $\frac{\partial L_1}{\partial \hat{\beta}} = 0$, we obtain the Bayes estimator as $\hat{\beta}_{SE} = \frac{n}{t}$

3.3. Estimation under El-Sayyad's loss

With El-Sayyad loss function, the corresponding Bayes estimator for β with respect to posterior distribution is given by

$$\begin{aligned} L_2 &= \int_0^\infty \beta^l (\hat{\beta}^r - \beta^r)^2 \frac{t^n}{\Gamma(n)} \beta^{n-1} e^{-t\beta} d\beta \\ L_2 &= \frac{\Gamma(l+n)}{\Gamma(n)t^l} \hat{\beta}^{2r} - 2\hat{\beta}^r \frac{\Gamma(r+l+n)}{\Gamma(n)t^{(r+l)}} + \frac{\Gamma(2r+l+n)}{\Gamma(n)t^{2r+l+n}} \end{aligned}$$

Solving $\frac{\partial L_2}{\partial \hat{\beta}^r} = 0$, we get the Bayes estimator under El-Sayyad loss function as

$$\hat{\beta}_{ES} = \frac{1}{t} \left[\frac{\Gamma(r+l+n)}{\Gamma(l+n)} \right]^{1/r}$$

3.4. Estimation under linex loss

To obtain the Bayes estimator, we minimize the posterior expected loss given by

$$\begin{aligned} L_3 &= \int_0^\infty [\exp(-c\Delta) - c\Delta - 1] \frac{t^n}{\Gamma(n)} \beta^{n-1} e^{-t\beta} d\beta \\ L_3 &= \left[\frac{t}{t-c} \right]^n \exp[-c\hat{\beta}] - c\hat{\beta} + \frac{cn}{t} - 1 \end{aligned}$$

Solving $\frac{\partial L_3}{\partial \hat{\beta}} = 0$, we obtain the Bayes estimator as $\hat{\beta}_{LIN} = \frac{n}{c} \log\left(\frac{t}{t-c}\right)$

4. PROBABILITY WEIGHTED MOMENT ESTIMATION

The probability weighted moment estimator for β, assuming that shape parameter α is known can be obtained as

$$\alpha_r = E[x(1 - F(x))^r]$$

We obtained probability weighted moment estimator of α, replacing α_0 by their unbiased estimate a_0 as

$$a_0 = n^{-1} \sum_{i=1}^{n} \frac{\binom{n-i}{r} x_i}{\binom{n-1}{r}}, \quad r = 0, 1, 2, ..., n-1$$

For the Frechet distribution, the probability weighted moment estimator for β is

$$\hat{\beta}_{PWM} = \frac{\overline{x}}{\Gamma((\alpha-1)/\alpha)}, \alpha > 1, \qquad \overline{x} = \frac{\sum x_i}{n}$$

Table 1 : Comparison between MLE, Bayesian and PWM when α =1.5, β =0.5, r =1, l =1

n	$\hat{\beta}_{ML}$	$\hat{\beta}_{SE}$	$\hat{\beta}_{PWM}$	$\hat{\beta}_{ES}$	$\hat{\beta}_{LIN}(c=1)$	$\hat{\beta}_{LIN}(c=-1)$
5	0.5627	0.4380	0.4898	0.5256	0.4648	0.4159
	(0.0370)	(0.0533)	(0.5844)	(0.0721)	(0.0668)	(0.0464)
10	0.5302	0.3934	0.4873	0.4327	0.4024	0.3850
	(0.0155)	(0.0307)	(0.2388)	(0.0279)	(0.0310)	(0.0308)
20	0.5109	0.3683	0.4998	0.3867	0.3719	0.3648
	(0.0060)	(0.0244)	(0.2839)	(0.0206)	(0.0238)	(0.0251)
30	0.5093	0.3655	0.4801	0.3776	0.3678	0.3632
	(0.0039)	(0.0226)	(0.1044)	(0.0198)	(0.0222)	(0.0231)
50	0.5050	0.3600	0.4957	0.3672	0.3614	0.3587
	(0.0022)	(0.0221)	(0.0874)	(0.0202)	(0.0217)	(0.0224)
100	0.5036	0.3580	0.5129	0.3615	0.3586	0.3573
	(0.0011)	(0.0214)	(0.7945)	(0.0204)	(0.0212)	(0.0216)

5. SIMULATION STUDY

To assess the performance of the methods, we generated N=1000 samples of sizes n = 5, 10, 20, 30, 50 and 100 from Frechet distribution [α= (1.5, 2, 3) and β= (0.5, 1)]. The value of the shape parameter is started from α=1.5, because for α=1, the PWM estimator does not exist. All possible combinations of the parameters have been considered. The averages of these estimates and the corresponding mean square errors (within parenthesis) were calculated for each method and presented in Tables 1 to 6 for comparison purpose.

6. CONCLUSION

Comparisons are made between the different estimators based on simulation study and effect of symmetric and asymmetric loss functions respect to various sample size and we observed the following:

1. In general, the ML estimator performs better than other estimators in terms of biases for all cases considered. Whereas MSE decreases for PWM method with increasing α.
2. It is also concluded that Bayes estimates based on squared error loss function and El-Sayyad function are very close to the ML estimator for β=1 and different values of α as sample size increases. Moreover, Bayes estimate relative to the linex loss function is also close to the ML estimate for the case when c= -1 and β=1. We also concluded that Bayes estimate under linex loss function for c=1 and β=1 is confining to the ML estimate as sample size increases. As in the case of different values of r, l and c, we obtain approximately the same results. Finally, we can say that in each scenario,

the ML method outperforms in terms of bias and MSE.

Table 2: Comparison between MLE, Bayesian and PWM when α =2, β =0.5, r =1, l =1

n	$\hat{\beta}_{ML}$	$\hat{\beta}_{SE}$	$\hat{\beta}_{PWM}$	$\hat{\beta}_{ES}$	$\hat{\beta}_{LIN}(c=1)$	$\hat{\beta}_{LIN}(c=-1)$
5	0.5428	0.3142	0.5066	0.3770	0.3289	0.3018
	(0.0214)	(0.0692)	(0.1487)	(0.0651)	(0.0761)	(0.0671)
10	0.5179	0.2761	0.5008	0.3038	0.2806	0.2719
	(0.0082)	(0.0599)	(0.0535)	(0.0504)	(0.0587)	(0.0612)
20	0.5118	0.2657	0.4964	0.2790	0.2676	0.2638
	(0.0038)	(0.0591)	(0.0256)	(0.0535)	(0.0583)	(0.0598)
30	0.5076	0.2599	0.5086	0.2686	0.2611	0.2587
	(0.0023)	(0.0600)	(0.0486)	(0.0561)	(0.0595)	(0.0605)
50	0.5043	0.2556	0.4968	0.2608	0.2563	0.2550
	(0.0013)	(0.0610)	(0.0095)	(0.0586)	(0.0607)	(0.0614)
100	0.5011	0.2517	0.5060	0.2542	0.2520	0.2514
	(0.0006)	(0.0622)	(0.0112)	(0.0610)	(0.0620)	(0.0624)

Table 3: Comparison between MLE, Bayesian and PWM when α =3, β =0.5, r =1, l =1

n	$\hat{\beta}_{ML}$	$\hat{\beta}_{SE}$	$\hat{\beta}_{PWM}$	$\hat{\beta}_{ES}$	$\hat{\beta}_{LIN}(c=1)$	$\hat{\beta}_{LIN}(c=-1)$
5	0.5210	0.1529	0.4975	0.1835	0.1561	0.1500
	(0.0074)	(0.1275)	(0.0195)	(0.1104)	(0.1261)	(0.1289)
10	0.5123	0.1393	0.4964	0.1532	0.1404	0.1382
	(0.0032)	(0.1323)	(0.0088)	(0.1229)	(0.1316)	(0.1330)
20	0.5056	0.1317	0.5034	0.1383	0.1322	0.1313
	(0.0016)	(0.1366)	(0.0060)	(0.1319)	(0.1363)	(0.1370)
30	0.5042	0.1296	0.4995	0.1339	0.1299	0.1293
	(0.0009)	(0.1377)	(0.0031)	(0.1346)	(0.1375)	(0.1379)
50	0.5018	0.1272	0.4982	0.1297	0.1274	0.1270
	(0.0005)	(0.1392)	(0.0022)	(0.1374)	(0.1391)	(0.1394)
100	0.5017	0.1267	0.5002	0.1280	0.1268	0.1266
	(0.0002)	(0.1394)	(0.0010)	(0.1385)	(0.1394)	(0.1395)

Table 4: Comparison between MLE, Bayesian and PWM when α =1.5, β =1, r =1, l =1

n	$\hat{\beta}_{ML}$	$\hat{\beta}_{SE}$	$\hat{\beta}_{PWM}$	$\hat{\beta}_{ES}$	$\hat{\beta}_{LIN}(c=1)$	$\hat{\beta}_{LIN}(c=-1)$
5	1.1238	1.2469	0.9338	1.4963	1.4002	1.0828
	(0.1835)	(0.6107)	(1.2068)	(1.0380)	(0.6202)	(0.2912)
10	1.0611	1.1060	0.9255	1.2166	1.1835	1.0428
	(0.0590)	(0.1742)	(0.4822)	(0.2442)	(0.2762)	(0.1240)
20	1.0315	1.0566	0.9425	1.1094	1.0872	1.0283
	(0.0257)	(0.0637)	(0.6156)	(0.0787)	(0.0761)	(0.0547)
30	1.0176	1.0322	0.9670	1.0666	1.0510	1.0142
	(0.0153)	(0.0367)	(0.4259)	(0.0425)	(0.0411)	(0.0334)
50	1.0078	1.0149	0.9460	1.0352	1.0256	1.0046
	(0.0086)	(0.0199)	(0.2034)	(0.0217)	(0.0211)	(0.0189)
100	1.0046	1.0086	0.9920	1.0187	1.0138	1.0035
	(0.0044)	(0.0101)	(0.1576)	(0.0106)	(0.0104)	(0.0098)

Table 5: Comparison between MLE, Bayesian and PWM when α =2, β =1, r =1, l =1

n	$\hat{\beta}_{ML}$	$\hat{\beta}_{SE}$	$\hat{\beta}_{PWM}$	$\hat{\beta}_{ES}$	$\hat{\beta}_{LIN}(c=1)$	$\hat{\beta}_{LIN}(c=-1)$
5	1.0677	1.2083	1.0034	1.4500	1.7160	1.0578
	(0.0728)	(0.4701)	(0.3904)	(0.8170)	(0.5928)	(0.2345)
10	1.0404	1.1116	1.0213	1.2228	1.1877	1.0484
	(0.0308)	(0.1544)	(0.3856)	(0.2214)	(0.2254)	(0.2254)
20	1.0167	1.0468	0.9845	1.0991	1.0768	1.0190
	(0.0132)	(0.0603)	(0.0965)	(0.0739)	(0.0716)	(0.0522)
30	1.0095	1.0273	0.9983	1.0616	1.0459	1.0096
	(0.0083)	(0.0357)	(0.0894)	(0.0411)	(0.0398)	(0.0326)
50	1.0090	1.0232	0.9993	1.0436	1.0340	1.0126
	(0.0051)	(0.0216)	(0.0485)	(0.0238)	(0.0231)	(0.0203)
100	1.0047	1.0120	1.0017	1.0221	1.0172	1.0069
	(0.0026)	(0.0108)	(0.0796)	(0.0113)	(0.0112)	(0.0105)

Table 6: Comparison between MLE, Bayesian and PWM when α =3, β =1, r =1, l =1

n	$\hat{\beta}_{ML}$	$\hat{\beta}_{SE}$	$\hat{\beta}_{PWM}$	$\hat{\beta}_{ES}$	$\hat{\beta}_{LIN}(c=1)$	$\hat{\beta}_{LIN}(c=-1)$
5	1.0461	1.2462	0.9892	1.4954	1.1278	1.0843
	(0.0327)	(0.5658)	(0.0833)	(0.9729)	(0.9530)	(0.2741)
10	1.0216	1.1045	0.9990	1.2149	1.1803	1.0418
	(0.0125)	(0.1590)	(0.0456)	(0.2254)	(0.2365)	(0.1158)
20	1.0139	1.0601	1.0058	1.1131	1.0909	1.0316
	(0.0059)	(0.0621)	(0.0263)	(0.0773)	(0.0742)	(0.0533)
30	1.0092	1.0407	1.0006	1.0754	1.0600	1.0224
	(0.0043)	(0.0435)	(0.0148)	(0.0503)	(0.0487)	(0.0394)
50	1.0046	1.0211	1.0049	1.0415	1.0319	1.0106
	(0.0023)	(0.0226)	(0.0083)	(0.0248)	(0.0241)	(0.0213)
100	1.0010	1.0064	1.0008	1.0164	1.0115	1.0013
	(0.0011)	(0.0105)	(0.0047)	(0.0109)	(0.0108)	(0.0102)

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