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Objective Bayesian analysis of Pareto distribution under progressive Type-II censoring☆

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ABSTRACT

In this paper, objective Bayesian analysis of Pareto distributions under progressive Type-II censoring is considered. Three types of noninformative priors (Jeffreys prior, two reference priors and two general forms of second order probability matching prior) are given. We show that one of the reference priors is also the second order probability matching prior (PMP). However, this reference prior, as well as the Jeffreys prior and PMPs, are related to the censoring scheme of life tests. Thus the other reference prior is preferred, and the proprieties of the posterior are discussed based on it. The simulation shows that the second reference prior leads to the required frequentist property. A real data from Crowley and Hu (1977) is analyzed for the purpose of illustration.

1. Introduction

Progressive censoring, more flexible than the ordinary censoring (Herd, submitted for publication), has now become one of the most popular censoring schemes in life testing. In recent literature about progressive censoring, both classical and subjective Bayesian approaches are widely utilized in parameter estimation of various distributions. Cohen (1963) considered the maximum likelihood estimates of the parameters when the samples are from normal or exponential distributions under progressive censoring. Kundu (2008) and Kim et al. (2011) discussed the Bayesian estimation for the Weibull model and the exponentiated Weibull model under the Type-II progressive censoring, respectively; and both of these focused on the conjugate priors for the parameters. For the case that the observed data are from the Pareto distribution under the progressive Type-II censoring, Amin (2008) and Soliman (2008) considered the estimation and prediction issues using the subjective Bayesian approach; the main difference between them is choosing different priors for the parameters. For a comprehensive recent review of progressive censoring, see Balakrishnan (2007).

The above references are either from the classical perspective or from the subjective Bayesian perspective. However, for the classical maximum likelihood method, numerical techniques are required and the estimates can have good properties only when the sample size is large enough. And in some situations, the subjective Bayesian method may also not be practical because of the difficulty of determination of the prior distribution which is usually elicited based on the historical data or the experience of experts. See Berger (2006). With limited time and little knowledge about the prior information, the obtained subjective priors can be "quite bad" so that the sensitivity or robustness of the priors need to be considered. Instead, the objective Bayesian approach is an alternative way, which has become more and more popular and acceptable due to the continuous work of some statisticians. Therefore, we focus on the objective Bayesian approach. For objective

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Bayes, the Jeffreys prior and the reference prior are the two most used noninformative priors. The Jeffreys prior (Jeffreys, 1961) is the square root of the determinant of the Fisher information matrix, and performs well in one dimensional parameter models. However, Jeffreys himself noticed difficulties of the method for multi-dimensional cases, and would provide *ad hoc* modifications to the prior. Bernardo (1979) proposed the reference prior to overcome the drawbacks of the Jeffreys prior in multi-dimensional settings. For more details, see Berger and Bernardo (1989, 1992). Moreover, due to its frequentist properties (the corresponding posterior probabilities of certain regions coincide with their frequentist coverage probabilities), the probability matching prior (PMP) is another popular noninformative prior in objective Bayesian analysis. See Datta and Sweeting (2005).

In this paper, we consider the objective Bayesian framework for the analysis of the Pareto distribution based on the progressive Type-II censoring. The Jeffreys prior and two reference priors of the parameters are derived. Besides, two general forms of the second order PMP are given, and we show that one of the reference priors is also a PMP. See Section 3. However, the Jeffreys prior and the PMPs depend on the type of censoring scheme in the life testing, which seems not so reasonable in objective Bayesian philosophy. For this reason, we recommend the other reference prior, although this is not a PMP, to proceed with the posterior analysis. Based on this reference prior, the proprieties of the posterior are discussed in Section 4, and the frequentist coverage probabilities of the α quantiles of the posterior marginal distributions of the parameters are computed via a numerical simulation in Section 5. The simulation shows that the reference prior performs well. Finally, real data from Crowley and Hu (1977) are analyzed for illustration.

2. Model

Assume that the data come from the Pareto distribution with scale parameter θ and shape parameter ν. The probability density function and cumulative distribution function are given by

$$f(x;\theta,\nu)=\nu\theta^{\nu}x^{-(\nu+1)} \tag{1}$$

and

$$F(x;\theta,\nu)=1-\left(\frac{\theta}{x}\right)^{\nu},\quad x>\theta,\nu>0,\theta>0, \tag{2}$$

respectively. Under progressive Type-II right censoring, assume that there are n identical units on test. Once the first failure occurs, a number R_1 of surviving units are removed randomly. Next, following the second observed failure, R_2 surviving units are removed at random from the $n-R_1-1$ units left, and so on. Finally, at the time of the mth observed failure, all the remaining $R_m=n-R_1-\cdots-R_{m-1}-m$ surviving units are removed from the test. In this situation, the number of failures $m>0$ and the progressive censoring scheme $(R_1,\ldots,R_m)$ are all assumed to be pre-fixed. Specially, when $R_1=\cdots=R_{m-1}=0$, $0<R_m<n$, it is the ordinary Type-II censoring.

Let $x_1,x_2,\ldots,x_m$ be the failure times observed. Then from Balakrishnan (2007), we have the likelihood function

$$\begin{aligned}L(\theta,\nu|x_1,x_2,\ldots,x_m)&=C\prod_{i=1}^{m}f(x_i)(1-F(x_i))^{R_i}\\&=C\prod_{i=1}^{m}\nu\theta^{\nu(R_i+1)}x_i^{-\nu(R_i+1)-1}\\&=C\nu^m\theta^{\nu n}\prod_{i=1}^{m}x_i^{-\nu(R_i+1)-1},\end{aligned} \tag{3}$$

where $x_1\le\cdots\le x_m$ and $C=n(n-R_1-1)\cdots(n-R_1-\cdots-R_{m-1}-m+1)$.

3. Noninformative priors

From (3), we obtain the Fisher information matrix of (θ,ν)

$$I(\theta,\nu)=\begin{pmatrix}\dfrac{n\nu}{\theta^2} & -\dfrac{n}{\theta}\\ -\dfrac{n}{\theta} & \dfrac{m}{\nu^2}\end{pmatrix}. \tag{4}$$

Thus the Jeffreys prior is

$$\pi_J(\theta,\nu)\propto\frac{1}{\theta}\sqrt{\frac{m}{\nu}-n},$$

where $\nu<m/n$ is a constraint which may not be satisfied in practice.

Reference priors are related to the parameter of interest. That is, different orderings of parameters of interest will lead to different reference priors. For the orders of (θ,ν) and (ν,θ), we have the following theorem.

Theorem 3.1. (a) *When θ is the parameter of interest, the reference prior of (θ, ν) is*

$$\pi_{R_1}(\theta, \nu) \propto \frac{1}{\theta\nu}.$$

(b) *When ν is the parameter of interest, the reference prior of (ν, θ) is*

$$\pi_{R_2}(\nu, \theta) \propto \frac{\sqrt{m - n\nu}}{\theta\nu}, \quad \text{for } \nu < m/n.$$

Proof. For (a), using the procedure in Berger and Bernardo (1989), we choose a sequence of compact sets $\Omega_i = (a_{1i}, a_{2i}) \times (b_{1i}, b_{2i})$ for (θ, ν) such that $a_{1i}, b_{1i} \to 0$ and $a_{2i} \to x_1, b_{2i} \to \infty$ as $i \to \infty$. Then in Ω_i, given θ, the conditional prior of ν is

$$\pi_i(\nu|\theta) = \frac{1}{\nu(\log b_{2i} - \log b_{1i})}\mathbf{1}_{(b_{1i}, b_{2i})}(\nu),$$

where $\mathbf{1}_A$ denotes the indicator of a set A. The marginal prior for θ is

$$\begin{aligned}\pi_i(\theta) &= \exp\left\{\frac{1}{2}\int_{b_{1i}}^{b_{2i}} \pi_i(\nu|\theta)\log\left[\frac{(nm - n^2\nu)/(\theta^2\nu)}{m/\nu^2}\right]d\nu\right\}\\ &= \exp\left\{\frac{1}{2}\frac{1}{\log b_{2i} - \log b_{1i}}\int_{b_{1i}}^{b_{2i}} \frac{1}{\nu}\log(mn\nu - n^2\nu^2)d\nu\right\} \times \frac{1}{\sqrt{m\theta^2}}\\ &= \frac{C_i}{\theta},\end{aligned}$$

where C_i is a constant. Thus, the reference prior for (θ, ν) is

$$\pi_{R_1}(\theta, \nu) = \lim_{i\to\infty}\frac{\pi_i(\theta)\pi_i(\nu|\theta)}{\pi_i(\theta_0)\pi_i(\nu_0|\theta_0)} \propto \frac{1}{\theta\nu},$$

where $\theta_0 = \frac{x_1}{2}$ and $\nu_0 = 1$. Then the result follows.

For (b), a similar procedure can be implemented. Note that the conditional prior of θ in Ω_i is

$$\pi_i(\theta|\nu) = \frac{1}{\theta(\log a_{2i} - \log a_{1i})}\mathbf{1}_{(a_{1i}, a_{2i})}(\theta),$$

and the marginal prior for ν is

$$\begin{aligned}\pi_i(\nu) &= \exp\left\{\frac{1}{2}\int_{a_{1i}}^{a_{2i}} \pi_i(\theta|\nu)\log\left[\frac{(nm - n^2\nu)/(\theta^2\nu)}{n\nu/\theta^2}\right]d\theta\right\}\\ &= \exp\left\{\frac{1}{2}\log\left[\frac{m - n\nu}{\nu^2}\right]\right\}\\ &= \frac{\sqrt{m - n\nu}}{\nu}.\end{aligned}$$

Therefore, the reference prior for (ν, θ) is

$$\pi_{R_2}(\nu, \theta) = \lim_{i\to\infty}\frac{\pi_i(\nu)\pi_i(\theta|\nu)}{\pi_i(\nu_0)\pi_i(\theta_0|\nu_0)} \propto \frac{\sqrt{m - n\nu}}{\theta\nu},$$

where $\theta_0 = \frac{x_1}{2}$ and $\nu_0 = 1$. Then the result holds. □

From Theorem 3.1(b) we know that the reference prior $\pi_{R_2}(\nu, \theta)$ is also an inappropriate prior because of its undesirable constraint on ν.

Peers (1965) gave the formula to derive second order PMP for the two-parameter case. Let

$$I(\theta_1, \theta_2) = \begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$$

be the Fisher information matrix of (θ_1, θ_2) where θ_1 is of interest. Then π is a second order PMP for (θ_1, θ_2) if and only if π is the solution of the partial differential equation

$$\frac{\partial}{\partial\theta_2}\left(\frac{a_{12}\pi}{a_{22}\sqrt{K}}\right) - \frac{\partial}{\partial\theta_1}\left(\frac{\pi}{\sqrt{K}}\right) = 0, \tag{5}$$

where $K = a_{11} - a_{12}^2/a_{22}$. For the case considered here, we have the following theorem.

Theorem 3.2. (a) *When θ is the parameter of interest and ν is a nuisance parameter, the second order PMP is*

$$\pi_{M_1}(\theta,\nu) = \nu^{-3/2} f_1(\theta e^{\frac{m}{n\nu}}) e^{\frac{m}{n\nu}} \sqrt{m-n\nu}, \tag{6}$$

where f_1 is a continuously differentiable function.

(b) *When ν is the parameter of interest and θ is a nuisance parameter, the second order PMP is*

$$\pi_{M_2}(\nu,\theta) = \nu^{-2} f_2\left(\frac{\theta}{\nu}\right) \sqrt{m-n\nu}, \tag{7}$$

where f_2 is a continuously differentiable function.

(c) *The reference prior $\pi_{R_1}(\theta,\nu)$ is not a second order PMP, however, $\pi_{R_2}(\nu,\theta)$ is a second order PMP.*

Proof. For (a), according to the partial differential equation (5), we have $a_{11} = \frac{n\nu}{\theta^2}$, $a_{12} = -\frac{n}{\theta}$, $a_{22} = \frac{m}{\nu^2}$ and $K = \frac{mn\nu - n^2\nu^2}{m\theta^2}$. The second order probability matching prior $\pi_{M_1}(\theta,\nu)$ must satisfy

$$\frac{\partial}{\partial\nu}\left(\frac{n\nu^2\pi_{M_1}(\theta,\nu)}{m\sqrt{\nu(m-n\nu)}}\right) + \frac{\partial}{\partial\theta}\left(\frac{\theta\pi_{M_1}(\theta,\nu)}{\sqrt{\nu(m-n\nu)}}\right) = 0.$$

The solution of the differential equation is given by formula (6).

For (b), similarly, the second order probability matching prior $\pi_{M_2}(\theta,\nu)$ must satisfy

$$\frac{\partial}{\partial\nu}\left(\frac{\nu\pi_{M_2}(\nu,\theta)}{\sqrt{m-n\nu}}\right) + \frac{\partial}{\partial\theta}\left(\frac{\theta\pi_{M_2}(\nu,\theta)}{\sqrt{m-n\nu}}\right) = 0.$$

The solution of the above differential equation is given by formula (7).

For (c), when θ is the parameter of interest,

$$\begin{aligned}\frac{\partial}{\partial\nu}\left(\frac{a_{12}\pi_{R_1}}{a_{22}\sqrt{K}}\right) - \frac{\partial}{\partial\theta}\left(\frac{\pi_{R_1}}{\sqrt{K}}\right) &= -\sqrt{\frac{m}{n}}\left(\frac{\partial}{\partial\nu}\frac{n\nu^2\pi_{R_1}(\theta,\nu)}{m\sqrt{\nu(m-n\nu)}} + \frac{\partial}{\partial\theta}\frac{\theta\pi_{R_1}(\theta,\nu)}{\sqrt{\nu(m-n\nu)}}\right)\\ &= -\sqrt{\frac{m}{n}}\left(\frac{\partial}{\partial\nu}\frac{n\nu/\theta}{m\sqrt{\nu(m-n\nu)}} + 0\right) \neq 0.\end{aligned}$$

That is, $\pi_{R_1}(\theta,\nu)$ does not satisfy the Eq. (5) so that it is not a second order PMP. On the other hand, when ν is the parameter of interest, taking $f_2(x) = \frac{1}{x}$ leads to $\pi_{R_2}(\nu,\theta)$. Then the result holds. □

Remarks.
- It can also be verified that the Jeffreys prior is not a PMP.
- The Jeffreys prior, the reference prior $\pi_{R_2}(\nu,\theta)$ and the PMPs are related to the censoring scheme of the tests and set an undesirable constraint on parameter ν, i.e. $\nu < m/n$. Therefore, for different censoring schemes, these priors will be different and the constraint for ν also changes. For example, assume that the true value $\nu = 1$ and that under progressive Type-II censoring, $n = 20$ and $m = 10$. If these noninformative priors are utilized, the parameter ν has a constraint: $\nu < 0.5$. Then the posterior estimate of ν will always underestimate the parameter ν regardless of the form of the observed failure times. As a result, these priors are ineffective in objective Bayesian analysis.
- The reference prior $\pi_{R_1}(\nu,\theta)$ is recommended. In the following sections, we will use $\pi_{R_1}(\nu,\theta)$ to proceed with the posterior analysis.

4. Posterior analysis

Given the likelihood function (3) and the reference prior $\pi_{R_1}(\theta,\nu)$, the joint posterior density of (θ,ν) can be obtained as

$$\begin{aligned}\pi_{R_1}(\theta,\nu|x_1,\ldots,x_m) &\propto L(\theta,\nu|x_1,\ldots,x_m)\pi_{R_1}(\theta,\nu)\\ &\propto \nu^m\theta^{\nu n}\exp\left\{-\nu\sum_{i=1}^m (R_i+1)\log x_i\right\}\times\frac{1}{\theta\nu}\\ &\propto \nu^{m-1}\theta^{\nu n-1}\exp\left\{-\nu\sum_{i=1}^m (R_i+1)\log x_i\right\}.\end{aligned} \tag{8}$$

Then we have the following result.

Theorem 4.1. *If $m > 1$ and $x_1,\ldots,x_m$ are not mutually equal, the posterior distribution of (θ,ν) based on $\pi_{R_1}(\theta,\nu)$ is proper.*

Proof. If $m > 1$ and $x_1, \ldots, x_m$ are not mutually equal, given the posterior density $\pi_{R_1}(\theta, \nu|x_1, \ldots, x_m)$ we have

$$\begin{aligned}
&\int_0^\infty \int_0^{x_1} \nu^{m-1}\theta^{\nu n-1} \exp\left\{-\nu \sum_{i=1}^m (R_i+1)\log x_i\right\} d\nu d\theta \\
&= \int_0^\infty \frac{1}{n}\nu^{m-2} \exp\left\{-\nu\left(\sum_{i=1}^m (R_i+1)\log x_i - n\log x_1\right)\right\} d\nu \\
&= \frac{\Gamma(m-1)}{n\left(\sum_{i=1}^m (R_i+1)\log x_i - n\log x_1\right)^{m-1}} < \infty.
\end{aligned}$$

Thus the result holds. □

Remarks. • If $m = 1$, noting that $R_1 = n-1$ and $\pi_{R_1}(\theta, \nu|x_1, \ldots, x_m) \propto \theta^{\nu n-1}\exp\{-\nu n \log x_1\}$, we have

$$\int_0^\infty \int_0^{x_1} \theta^{\nu n-1}\exp\{-\nu n\log x_1\}\, d\nu d\theta = \int_0^\infty \frac{1}{\nu n} d\nu = \infty.$$

• If $x_1 = x_2 = \cdots = x_m$, noting that $\pi_{R_1}(\theta, \nu|x_1, \ldots, x_m) \propto \nu^{m-1}\theta^{\nu n-1}x_1^{-\nu n}$, we have

$$\int_0^\infty \int_0^{x_1} \nu^{m-1}\theta^{\nu n-1}x_1^{-\nu n} d\nu d\theta = \int_0^\infty \frac{1}{n}\nu^{m-2} d\nu = \infty.$$

Furthermore, given the joint posterior density (8), it is easy to derive the marginal posterior densities of θ and ν.

Theorem 4.2. *If $m > 1$ and $x_1, \ldots, x_m$ are not mutually equal, the marginal posterior of ν, say $p(\nu|data)$, is Gamma$(m-1, \sum_{i=1}^m (R_i+1)\log x_i - n\log x_1)$. The marginal posterior density function of θ is given by*

$$p(\theta|data) = \frac{1}{c\theta\left(\sum_{i=1}^m (R_i+1)\log x_i - n\log\theta\right)^m}, \tag{9}$$

where $0 < \theta < x_1$ and $c = \frac{1}{n(m-1)(\sum_{i=1}^m (R_i+1)\log x_i - n\log x_1)^{m-1}}$ is the normalizing constant.

Proof. The marginal posterior is derived by integrating out the nuisance parameter in the joint posterior density. Therefore,

$$\begin{aligned}
p(\nu|data) &= \int_0^{x_1} \pi_{R_1}(\theta, \nu|x_1, \ldots, x_m) d\theta \\
&\propto \int_0^{x_1} \nu^{m-1}\theta^{\nu n-1}\exp\left\{-\nu\sum_{i=1}^m (R_i+1)\log x_i\right\} d\theta \\
&\propto \nu^{m-2}\exp\left\{-\nu\left(\sum_{i=1}^m (R_i+1)\log x_i - n\log x_1\right)\right\},
\end{aligned}$$

which leads to the first result. For θ, the marginal posterior

$$\begin{aligned}
p(\theta|data) &\propto \int_0^\infty \nu^{m-1}\theta^{\nu n-1}\exp\left\{-\nu\sum_{i=1}^m (R_i+1)\log x_i\right\} d\nu \\
&\propto \frac{1}{\theta\left(\sum_{i=1}^m (R_i+1)\log x_i - n\log\theta\right)^m}.
\end{aligned}$$

The normalizing constant is

$$\begin{aligned}
c &= \int_0^{x_1} \frac{1}{\theta\left(\sum_{i=1}^m (R_i+1)\log x_i - n\log\theta\right)^m} d\theta \\
&= \frac{1}{n(m-1)\left(\sum_{i=1}^m (R_i+1)\log x_i - n\log x_1\right)^{m-1}}.
\end{aligned}$$

Then the second result holds immediately. □

5. Simulation and data analysis

5.1. Simulation studies

To obtain the posterior samples of (θ, ν), the following sampling procedure is implemented.

1. Draw ν from the *Gamma* distribution $Gamma(m-1, \sum_{i=1}^{m}(R_i+1)\log x_i - n\log x_1)$.
2. Given the posterior cumulative probability distribution of θ

$$F(\theta|data) = \frac{1}{c \times n(m-1)\left(\sum_{i=1}^{m}(R_i+1)\log x_i - n\log\theta\right)^{m-1}}, \quad 0 \le \theta \le x_1, \tag{10}$$

a posterior sample of θ can be obtained from

$$\theta = F^{-1}(u),$$

where u is a random sample generated from the uniform distribution $Unif(0, 1)$.

To assess the frequentist performance of the reference prior $\pi_{R_1}(\theta, \nu)$, let $\theta_{R_1}(\alpha; data)$ and $\nu_{R_1}(\alpha; data)$ be the posterior α quantile of θ and ν, respectively.

$$Q_{R_1}(\alpha; \theta) = P_{(\theta,\nu)}(0 \le \theta \le \theta_{R_1}(\alpha; data))$$

denotes a frequentist coverage probability of $\theta_{R_1}(\alpha; data)$ where $\theta_{R_1}(\alpha; data)$ is taken as a random variable. Similarly, $Q_{R_1}(\alpha; \nu)$ denotes the frequentist coverage probability of $\nu_{R_1}(\alpha; data)$.

Take $\theta = 1.5$, $\nu = 0.9$, $n = 12, 20, 100$, $m = 2, 3, 5, 7, 10, 15, 20$, and the pre-fixed scheme $R = (R_1, \ldots, R_m)$ shown in Table 1. For $\alpha = 0.95$ and 0.90, the coverage probabilities $Q_{R_1}(\alpha; \theta)$ and $Q_{R_1}(\alpha; \nu)$ are calculated, along with the coverage probabilities of the maximum likelihood estimates (MLEs) denoted as $Q_{\mathrm{MLE}}(\alpha; \cdot)$. In the parentheses are the corresponding average length of the confidence intervals. See Table 1. Table 1 was computed as follows.

1. For each censoring scheme, a number of 100,000 Type-II progressive censoring samples $(x_1, \ldots, x_m)$ are generated from the Pareto distribution with scale parameter $\theta = 1.5$ and shape parameter $\nu = 0.9$.
2. For each generated sample $(x_1, \ldots, x_m)$, the posterior α quantiles of θ and ν are obtained based on the procedure mentioned at the beginning of this section.
3. $Q_{R_1}(\alpha; \theta)$ is estimated by the proportion of the true value less than the posterior α quantiles of θ. $Q_{R_1}(\alpha; \nu)$ can be assessed similarly.
4. Since the Fisher information matrix is not a positive definite matrix when $\nu \ge m/n$, we cannot derive the confidence intervals of the MLE. For the purpose of comparison, however, we calculate $Q_{\mathrm{MLE}}(\alpha; \cdot)$ by taking the parameters (θ, ν) as orthogonal, i.e. taking the Fisher information matrix as diagonal. In this way, the confidence intervals of the MLE can be obtained through large sample theory; and $Q_{\mathrm{MLE}}(\alpha; \cdot)$ comes out consequently.

From Table 1, we can see that the frequentist coverage probabilities of $\theta_{R_1}(\alpha; data)$ and $\nu_{R_1}(\alpha; data)$, say $Q_{R_1}(\alpha; \theta)$ and $Q_{R_1}(\alpha; \nu)$, are close to α even for the case $m = 2$. However, $Q_{\mathrm{MLE}}(\alpha; \cdot)$ have great bias; and when m becomes larger, such frequentist probabilities of θ become 1.000 which is obviously unreasonable. Therefore, though $\pi_{R_1}(\theta, \nu)$ is not a PMP, it provides much better performance in meeting the target coverage probability and serves to estimate the parameters more accurately even when the sample size is very small.

5.2. A real data example

The Pareto distribution has been widely used in describing survival time. We now consider the following real example: the Stanford heart transplant program reported by Crowley and Hu (1977) and used in Hossain and Zimmer (2000) and Soliman (2008). Of the 20 patients, the first three are dropped off at the beginning of the program, and we only consider the rest 17 patients in our analysis. Moreover, six patients were withdrawn from the program randomly. Consequently, the example is based on a sample of $n = 17$ with $m = 11$ and $R = (1, 0, 0, 0, 1, 0, 0, 2, 0, 2, 0)$. Table 2 shows the observed data.

The log–log probability plot implies that the Pareto distribution fits the data quite well and can be used for our analysis. Based on the observations and the reference prior $\pi_{R_1}(\theta, \nu)$, the objective Bayesian estimates (OBE) of the parameters, as well as the survival function $S(t)$ when $t = 80$, under the squared error loss (i.e. posterior mean) are calculated in Table 3, along with their corresponding standard errors (SE) and 95% posterior confidence intervals (CI). In addition, the classical MLEs are also given in Table 3 for comparison. Furthermore, the posterior marginal densities of θ and ν are plotted as in Fig. 1.

According to the results presented in Table 3, we know that

1. the Bayesian estimate of θ is considerably better because the MLE is roughly the first order statistics x_1;
2. the standard errors of the objective Bayesian estimates are much smaller than that of maximum likelihood estimates and their 95% confidence intervals also indicate that the objective Bayesian estimates are much more accurate than MLEs.

Table 1
Frequentist coverage probabilities with $\alpha = 0.95, 0.90$.

Censoring scheme				$Q_{R_1}(\alpha;\cdot)$		$Q_{\mathrm{MLE}}(\alpha;\cdot)$	
n	m	$(R_1,\ldots,R_m)$		$\alpha = 0.95$	$\alpha = 0.90$	$\alpha = 0.95$	$\alpha = 0.90$
12	2	(5, 5)	θ	0.950	0.899	0.935	0.912
				(1.375)	(1.114)	(1.316)	(1.104)
			ν	0.949	0.900	0.991	0.987
				(40.965)	(32.924)	(61.988)	(52.022)
12	3	(3, 3, 3)	θ	0.951	0.900	0.984	0.972
				(0.966)	(0.743)	(1.611)	1.352
			ν	0.949	0.900	0.988	0.970
				(4.834)	(3.980)	(6.158)	(5.168)
12	5	(2, 1, 1, 1, 2)	θ	0.951	0.901	0.996	0.991
				(0.716)	(0.565)	(1.824)	(1.530)
			ν	0.950	0.901	0.980	0.929
				(2.302)	(1.915)	(2.628)	(2.205)
20	3	(6, 5, 6)	θ	0.952	0.902	0.992	0.985
				(0.684)	(0.500)	(1.203)	(1.009)
			ν	0.950	0.900	0.988	0.970
				(4.804)	(3.956)	(6.120)	(5.136)
20	5	(3, 3, 3, 3, 3)	θ	0.950	0.900	0.998	0.996
				(0.466)	(0.358)	(1.359)	(1.140)
			ν	0.951	0.901	0.981	0.928
				(2.296)	(1.910)	(2.621)	(2.200)
20	7	(2, 2, 2, 2, 1, 2, 2)	θ	0.950	0.900	0.999	0.998
				(0.407)	(0.319)	(1.420)	(1.191)
			ν	0.950	0.900	0.971	0.919
				(1.704)	(1.422)	(1.867)	(1.567)
100	5	(19, ..., 19)	θ	0.950	0.898	1.000	1.000
				(0.104)	(0.077)	(0.579)	(0.486)
			ν	0.950	0.900	0.980	0.928
				(2.307)	(1.919)	(2.633)	(2.210)
100	7	(13, ..., 13, 15)	θ	0.950	0.900	1.000	1.000
				(0.088)	(0.067)	(0.606)	(0.509)
			ν	0.950	0.900	0.972	0.917
				(1.706)	(1.423)	(1.869)	(1.568)
100	10	(9, ..., 9)	θ	0.951	0.902	1.000	1.000
				(0.079)	(0.062)	(0.625)	(0.525)
			ν	0.950	0.900	0.964	0.910
				(1.311)	(1.096)	(1.395)	(1.171)
100	15	(6, ..., 6, 4, 3)	θ	0.950	0.899	1.000	1.000
				(0.074)	(0.058)	(0.640)	(0.537)
			ν	0.950	0.900	0.959	0.906
				(1.009)	(0.845)	(1.051)	(0.882)
100	20	(4, 4, 4, ..., 4)	θ	0.949	0.899	1.000	1.000
				(0.071)	(0.056)	(0.647)	(0.543)
			ν	0.951	0.900	0.957	0.905
				(0.851)	(0.713)	(0.877)	(0.736)

Table 2
The observed data.

i	1	2	3	4	5	6	7	8	9	10	11
x_i	23	27	39	127	136	280	624	730	836	1024	1349
R_i	1	0	0	0	1	0	0	2	0	2	0

6. Conclusion

In this article, objective Bayesian analysis is presented to make inference about the progressive Type-II right censored sample of the Pareto distribution. The Jeffreys prior, two reference priors and the general forms of second order PMP are obtained. One of the reference priors $\pi_{R_2}(\theta,\nu)$ is a second order PMP. Since the Jeffreys prior, the reference prior $\pi_{R_2}(\theta,\nu)$ and the second order PMPs are associated with the censoring scheme, we recommend $\pi_{R_1}(\theta,\nu)$ as the prior to proceed with the posterior analysis and properties of the corresponding joint posterior are investigated. Although the reference prior $\pi_{R_1}(\theta,\nu)$ is not a second order PMP, the simulation shows that $\pi_{R_1}(\theta,\nu)$ performs great in meeting the target coverage

Table 3
Parameter estimates.

Parameter	OBE	SE	95% CI
θ	18.283	4.000	(8.185, 22.852)
ν	0.245	0.078	(0.118, 0.419)
$S(80)$	0.697	0.090	(0.491, 0.846)
	MLE	SE.	95% CI
θ	23	14.085	(6.925, 76.385)
ν	0.267	0.105	(0.061, 0.473)
$S(80)$	0.717	0.091	(0.537, 0.897)

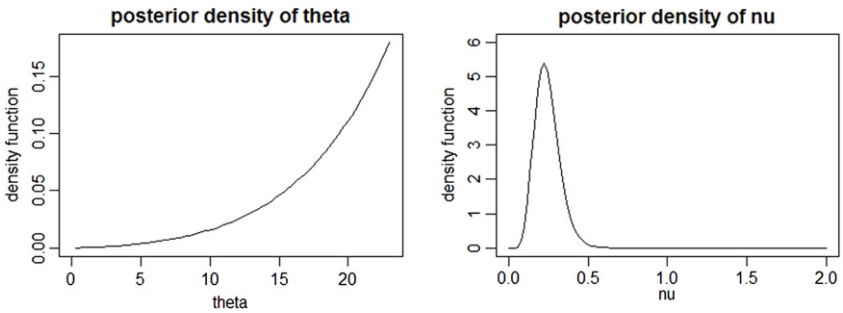

Fig. 1. Posterior marginal density.

probability. In the real data example where the sample size is not very large, we compare the posterior estimates based on $\pi_{R_1}(\theta, \nu)$ and the classical maximum likelihood estimates of the parameters. And we find that the objective Bayesian method outperforms the maximum likelihood approach.

In conclusion, objective Bayesian methodology has its superiority in specifying the prior distributions which can be appropriate and have good frequentist performance with small sample size.

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