Bayesian Analysis of Pareto Reliability With Dependent Masked Data

Ancha Xu and Yincai Tang

***Abstract*—In this paper, we consider a series system with two s-independent components, each of which has a Pareto distributed lifetime. We assume the lifetime data from the system are masked, and the masking probability is dependent. Noninformative prior, and conjugate prior are considered; and Gibbs sampling method is used to obtain the estimators of the unknown parameters, the reliability function, and the hazard function. A numeric example demonstrates how to apply the theoretical results.**

***Index Terms*—Bayesian estimators, Gibbs sampling, masking probability, Pareto distribution.**

ACRONYM[1]

PFM	Pareto failure model
MRS	The minimum random subset
FP	Flat prior, or noninformative prior
CP	Conjugate prior
FPN	Bayesian estimates with flat, and without masking
CPN	Bayesian estimates with conjugate priors without masking

NOTATION

$f(t\|\theta,\tau)$	pdf of the PFM; see (1)
θ,τ	parameters of the PFM; $\tau>0,\theta>0$
v	$\log(\tau)$
n	sample size
r	number of failures in a censored sample
t_i	the i-th order statistic from a sample of size n from the PFM, $i=1,2,\cdot,r$.
$t_{(11)}$	the observed minimal failure time from the first component
$t_{(21)}$	the observed minimal failure time from the second component
LT_r	total log time on test: $\sum_{i=1}^{r}\log(t_i)+(n-r)\log(t_r)$
$h(t),R(t)$	[hazard rate, reliability] of a series system with two s-independent components of the PFM at mission time t; see (2)
$f_j(t),R_j(t)$	[density function, reliability] of the j-th component of a series system of the PFM
$h_{\rm no}(t),R_{\rm no}(t)$	[hazard rate, reliability] without masking; see Tables III and IV
$B(1,p)$	Bernoulli distribution with success probability p
$\Gamma(a,b)$	Gamma distribution with shape parameter a, and scale parameter b
$GU(a,b,c)$	generalized uniform distribution with pdf: $f(x)=\frac{(a+1)x^a}{c^{a+1}-b^{a+1}}, b<x<c, a>0$
$Beta(\alpha,\beta)$	Beta distribution with parameters α, and β
S_i	the minimum random subset of the J possible failure causes of the system i
s_i	the realized set for S_i
K_i	the index of the component actually causing the i-th system to fail
$\boldsymbol{\varpi}$	all the parameters in the PFM
$\boldsymbol{\varpi}/\mu$	all the parameters except μ
$\hat{\phi}$	Bayesian estimate of ϕ with noninformation prior
$\tilde{\phi}$	Bayesian estimate of ϕ with conjugate prior

I. INTRODUCTION

IN masked system lifetime data, the exact failure causes are often unknown. Instead, a subset of components are responsible for the failures. The occurrence of masking is due to a variety of reasons such as lack of proper diagnostic equipment, cost and time constraints associated with failure analysis, recording errors, and the destructive nature of certain failed components that make exact diagnosis impossible.

Recently, several authors considered this problem for series systems assuming exponential or Weibull distributions for the component lifetimes. Most authors make an equiprobable assumption (called the independent assumption by some authors) for the masking probabilities where the conditional masking probabilities are equal given each cause of failure in the masked set. Miyakawa [10] considered the problem of a two-component series system when the system's components have constant failure rates. He derived closed-form expressions for the maximum likelihood estimators for the parameters based on the masked data. Usher & Hodgson [14] extended Miyakawa's

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[1]The singular and plural of an acronym are always spelled the same.

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results to a three-component series system under the same assumption. Doganaksoy [5] presented various methods to find confidence intervals for the three-component series systems with exponential-life components. Usher [15] considered the case of two-component series systems when the system's components' life times have Weibull distributions. Under the same assumption, Sarhan [11] obtained the maximum likelihood estimators for the parameters in the two-component, and three-component series systems. In some particular cases, he derived closed-form expressions for maximum likelihood estimates, which extended the results obtained by Usher & Hodgson [14].

However, the equiprobable assumption may not hold in some circumstances. To relax the equiprobable assumption, Lin & Guess [7], and Guttman *et al.* [6] consider a proportional probability model for a dependent masking probability for the two-component series system. Both assume that the masking probability conditional on the first component's failure and the system's failure time is proportional to that of the second component. Although both masking probabilities are functions of the system's failure time, the authors assume that the proportionality is s-independent of the system's failure time. Lynn & Tae [9] developed a Bayes approach to estimate parameters using the Gibbs sampler, and EM algorithm when the system's components' lifetimes have exponential or Weibull distributions. They explore two different stochastic models for the masking probabilities.

Except for exponential or Weibull distributions, the Pareto distribution also has important applications in reliability studies. The Pareto reliability model (PRM) arises as a mixture of exponential distributions, and its use for simulations with randomly varying environments is amply justified (Lindley [8]). Its probability density function is defined by

$$f(t|\theta,\tau)=\frac{\theta}{\tau}\left(\frac{\tau}{t}\right)^{\theta+1},t>\tau,\tau>0. \tag{1}$$

Hence, the reliability function, and hazard function of a series system with two independent components of the PFM at mission time t are

$$\begin{aligned} R(t)&=\left(\frac{\tau_1}{t}\right)^{\theta_1}\cdot\left(\frac{\tau_2}{t}\right)^{\theta_2},t>\max(\tau_1,\tau_2),\text{and}\\ h(t)&=\frac{\theta_1}{t}+\frac{\theta_2}{t}. \end{aligned} \tag{2}$$

Arnold & Press [2] presented Bayesian estimation for the PRM, when τ is known, and θ is unknown. When both τ and θ are unknown, Tiwari *et al.* [12] developed a fully Bayesian approach to estimate the parameters, and reliability function using the Gibbs sampler, and the rejection/acceptance algorithm. Upadhyay & Shastri [13]considered the full Bayesian analysis of the PRM when the observations are doubly censored. Ammar & Awad [1] considered the estimation of parameters of the PRM with a given τ when there are masked data. They compared the MLE with Bayesian estimation when making an equiprobable assumption. Basu *et al.*[3] presented a general parametric framework that is not restrictive in the number of potential causes of failure, their distributions, or even sampling schemes. However, they make an equiprobable assumption.

In reliability studies, the observations are usually censored to save time and cost. In this paper, we consider the case when the observations are masked, and censored. For unknown τ and θ, we present the Bayesian analysis of the PRM without the equiprobable assumption. In Section II, we state the effect of dependent masking on the likelihood, and the assumptions used in this paper. Bayesian inference of the PRM with dependent masked data is presented in Section III, where Gibbs sampling is used, and a numeric example is given to show the effectiveness of the method.

II. The Full Likelihood, and the Assumption

Assume we observe a sample of n series systems each having J s-independent components, and censored when the r-th sample fails. Let T_{ij} denote the lifetime of the j-th component in the i-th system with pdf $f_j(t)$, and survival function $R_j(t)$. Therefore, the lifetime of the i-th system is $T_i=\min(T_{i1},\ldots,T_{iJ})$, for $i=1,\ldots,n$. The censored observed data here are then $(t_1,s_1),\ldots,(t_r,s_r)$. Then the full likelihood for these data is

$$L_F=\prod_{i=r+1}^{n}P\{T_i>t_r\}\prod_{i=1}^{r}\sum_{j\in s_i}\left\{\left(f_j(t_i)\prod_{l=1,l\neq j}^{J}R_l(t_i)\right)P(S_i=s_i|T_i=t_i,K_i=j)\right\}. \tag{3}$$

We call $P(S_i=s_i|T_i=t_i,K_i=j)$ the masking probability, the conditional probability that the observed subset is s_i given that system i fails at time t_i, and the true cause is component j. Under an equiprobable assumption, that is, for a fixed j',

$$P(S_i=s_i|T_i=t_i,K_i=j')=P(S_i=s_i|T_i=t_i,K_i=j) \tag{4}$$

for all $j\in s_i$, (3) can be reduced to be

$$L_F=\prod_{i=1}^{r}\left[\sum_{j\in s_i}\left\{\left(f_j(t_i)\prod_{l=1,l\neq j}^{J}R_l(t_i)\right)\right\}\right]\times\prod_{i=r+1}^{n}P\{T_i>t_r\}. \tag{5}$$

In this paper, we only consider two-component systems ($J=2$), and assume the masking probabilities are s-independent of the failure time, but s-dependent on the component; that is, $P(S_i=\{1\}|T_i=t_i,K_i=1)=p_1$, and $P(S_i=\{2\}|T_i=t_i,K_i=2)=p_2$. Consequently, $P(S_i=\{1,2\}|T_i=t_i,K_i=1)=1-p_1$, and $P(S_i=\{1,2\}|T_i=t_i,K_i=2)=1-p_2$. Let r_1, and r_2 be the numbers of failed systems due to component 1, and 2 respectively; while r_{12} denotes the number of failed systems where the cause is masked. Besides, we assume T_{ij} have

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a Pareto distribution with parameters τ_j, and θ_j. Then the full likelihood is

$$\begin{aligned}L_F=&\prod_{i|s_i=\{1\}} p_1 f_1(t_i)R_2(t_i)\prod_{i|s_i=\{2\}} p_2 f_2(t_i)R_1(t_i)\\&\times\prod_{i|s_i=\{1,2\}}\{(1-p_1)f_1(t_i)R_2(t_i)+(1-p_2)f_2(t_i)R_1(t_i)\}\\&\times\prod_{i=r+1}^{n} P\{T_i>t_r\}\\=&\left(\prod_{i=1}^{r}R_1(t_i)R_2(t_i)\prod_{i=r+1}^{n}R_1(t_r)R_2(t_r)\right)\\&\times\prod_{i|s_i=\{1\}}p_1h_1(t_i)\prod_{i|s_i=\{2\}}p_2h_2(t_i)\\&\times\prod_{i|s_i=\{1,2\}}\{(1-p_1)h_1(t_i)+(1-p_2)h_2(t_i)\}\\=&(p_1\theta_1)^{r_1}(p_2\theta_2)^{r_2}\left[(1-p_1)\theta_1+(1-p_2)\theta_2\right]^{r_{12}}\\&\times\exp\{-\theta_1(LT_r-nv_1)\}\exp\{-\theta_2(LT_r-nv_2)\}\\&\times\exp\{-LT_r+(n-r)\log(t_r)\}.\end{aligned}\tag{6}$$

III. Bayesian Analysis

A. Gibbs Sampling

We introduce latent variables z_i for the systems of which the cause of failure is masked. Let z_i be a Bernoulli variable with a success probability of $\{z_i=1\}$ to be

$$\begin{aligned}P_i&=(1-p_1)h_1(t_i)/\left[(1-p_1)h_1(t_i)+(1-p_2)h_2(t_i)\right]\\&=(1-p_1)\theta_1/\left[(1-p_1)\theta_1+(1-p_2)\theta_2\right]\end{aligned}\tag{7}$$

for i in $s_i=\{1,2\}$.

Theorem 1: The probability of success P_i is exactly the probability that the true cause of failure of system i is due to the first component conditional on $s_i=\{1,2\}$, and t_i being observed; that is, $P_i=P(K_i=1|s_i,t_i)$.

The proof for the above theorem is given in the Appendix.

Let $z=\sum_{i|s_i=\{1,2\}} z_i$. Then the $[(1-p_1)\theta_1+(1-p_2)\theta_2]^{r_{12}}$ term in (6) can be simplified to

$$\left[(1-p_1)\theta_1+(1-p_2)\theta_2\right]^{r_{12}}\propto\left[(1-p_1)\theta_1\right]^{z}\left[(1-p_2)\theta_2\right]^{r_{12}-z}\tag{8}$$

From (6) and (8), and some simple algebra calculations, the conditional probability of the unknown parameters given data, and the latent variables z, is

$$\begin{aligned}&f(\varpi|data,z)\\&\quad\propto p_1^{r_1}(1-p_1)^{z}p_2^{r_2}(1-p_2)^{r_{12}-z}\theta_1^{(r_1+z)}\theta_2^{(r_2+r_{12}-z)}\\&\quad\times\exp\{-\theta_1(LT_r-nv_1)\}\exp\{-\theta_2(LT_r-nv_2)\}\pi(\varpi)\end{aligned}\tag{9}$$

where $\pi(\boldsymbol{\varpi})$ is the joint prior distribution. The conjugate priors of p_i, and θ_i are Beta, and Gamma densities respectively. Tiwari *et al.* [12] suggest the prior distribution of $\log(\tau_i)$ to be $\Gamma(a,b)$, or $\tau_i\propto constant$. However, this prior distribution is conjugate only when $b>n\theta$. Noticed that τ is a scale parameter. We use a non-information prior in this paper, $\tau_i\propto 1/\tau_i$ or $\log(\tau_i)\propto constant, i=1,2$.

TABLE I
The Simulated Data

Set of labels	System's failure time
$s_i=\{1\}$	5.160, 5.310, 5.324 , 5.431, 5.645, 6.217, 6.307
$s_i=\{2\}$	6.522, 6.867, 6.881 , 6.941, 6.966
$s_i=\{1,2\}$	7.259, 5.613, 6.842, 6.569, 6.074, 6.182, 6.187 , 7.215

TABLE II
The Bayesian Estimates of Parameters

Parameter	FP	CP	FPN	CPN
θ_1	1.2012	1.2176	1.168	1.190
θ_2	1.6820	1.7120	1.574	1.596
τ_1	5.2080	5.2134	5.054	5.054
τ_2	6.0938	6.0969	5.972	5.973
p_1	0.7297	0.7226		
p_2	0.4885	0.4848		

Now we describe the Gibbs sampling steps.

1) Given $(\theta_i,p_i), i=1,2$, generate $z_i\sim B(1,(1-p_1)\theta_1/[(1-p_1)\theta_1+(1-p_2)\theta_2])$ independently for i in $s_i=\{1,2\}$. Then compute $z=\sum_{i|s_i=\{1,2\}} z_i$.
2) Given the data, and z, update the parameters by

$$\begin{aligned}&p_1|LT_r,\varpi/p_1\sim Beta(r_1+\alpha_1,z+\beta_1)\\&p_2|LT_r,\varpi/p_2\sim Beta(r_2+\alpha_2,\beta_2+r_{12}-z)\\&\theta_1|LT_r,\varpi/\theta_1\sim\Gamma(a_1+r_1+z,b_1+LT_r-nv_1)\\&\theta_2|LT_r,\varpi/\theta_2\sim\Gamma(a_z,b_2+LT_r-nv_2)\\&\tau_1|LT_r,\varpi/\tau_1\sim GU(n\theta_1-1,0,t_{(11)})\\&\tau_2|LT_r,\varpi/\tau_2\sim GU(n\theta_2-1,0,t_{(21)})\end{aligned}\tag{10}$$

where $a_z=a_2+r_2+r_{12}-z$. The Gibbs sampler is implemented using the conditional posterior pdf (10).

B. Alternative Priors

The priors of p_1, p_2, θ_1, and θ_2 are the same as in subsection A. Choose the conjugate prior of τ_i conditioned on θ_i to be $\tau_i|\theta_i\propto\theta_i d_i^{-c_i\theta_i}\tau_i^{c_i\theta_i-1}, i=1,2$.

Then the Gibbs steps are as follows.

1) Given $(\theta_i,p_i), i=1,2$, generate $z_i\sim B(1,(1-p_1)\theta_1/[(1-p_1)\theta_1+(1-p_2)\theta_2])$ independently for i in $s_i=\{1,2\}$. Then compute $z=\sum_{i|s_i=\{1,2\}} z_i$.
2) Given the data, and z, we update the parameters by

$$\begin{aligned}&p_1|LT_r,\varpi/p_1\sim Beta(r_1+\alpha_1,z+\beta_1)\\&p_2|LT_r,\varpi/p_2\sim Beta(r_2+\alpha_2,\beta_2+r_{12}-z)\\&\theta_1|LT_r,\varpi/\theta_1\sim\Gamma(a_1+r_1+z+1,k_1)\\&\theta_2|LT_r,\varpi/\theta_2\sim\Gamma(a_2+r_1+r_{12}-z+1,k_2)\\&k_i=LT_r+b_i+c_i\log(d_i)-(n+c_i)v_i, i=1,2\\&\tau_1|LT_r,\varpi/\tau_1\sim GU\left((n+c_1)\theta_1-1,0,t_{(11)}\right)\\&\tau_2|LT_r,\varpi/\tau_2\sim GU\left((n+c_2)\theta_2-1,0,t_{(21)}\right)\end{aligned}\tag{11}$$

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TABLE III
POSTERIOR MEANS, AND LENGTHS OF 90% CONFIDENCE INTERVALS FOR $R(t)$

t	True R(t)	$\hat{R}(t)$	Length	$\hat{R}_{no}(t)$	Length
7	.5218	.5578	.1429	.5347	.1462
8	.3590	.3812	.1531	.3694	.1540
9	.2581	.2730	.1465	.2682	.1472
10	.1922	.2029	.1344	.1994	.1342
15	.0617	.0655	.0751	.0649	.0773
20	.0276	.0298	.0440	.0289	.0470
t	True R(t)	$\tilde{R}(t)$	Length	$\tilde{R}_{no}(t)$	Length
7	.5218	.5541	.1464	.5242	.1462
8	.3590	.3765	.1544	.3613	.1549
9	.2581	.2683	.1484	.2604	.1467
10	.1922	.1985	.1356	.1942	.1329
15	.0617	.0632	.0740	.0618	.0742
20	.0276	.0284	.0431	.0271	.0442

TABLE IV
POSTERIOR MEANS, AND LENGTHS OF 90% CONFIDENCE INTERVALS FOR $h(t)$

t	True h(t)	$\hat{h}(t)$	Length	$\hat{h}_{no}(t)$	Length
7	.4000	.4119	.1778	.3918	.1665
8	.3500	.3604	.1556	.3428	.1457
9	.3111	.3203	.1382	.3047	.1295
10	.2800	.2883	.1245	.2743	.1166
15	.1867	.1922	.0830	.1828	.0777
20	.1400	.1442	.0622	.1371	.0582
t	True h(t)	$\tilde{h}(t)$	Length	$\tilde{h}_{no}(t)$	Length
7	.4000	.4185	.1843	.3980	.1665
8	.3500	.3662	.1612	.3483	.1457
9	.3111	.3255	.1433	.3096	.1295
10	.2800	.2930	.1289	.2786	.1165
15	.1867	.1953	.0860	.1857	.0777
20	.1400	.1465	.0645	.1393	.0583

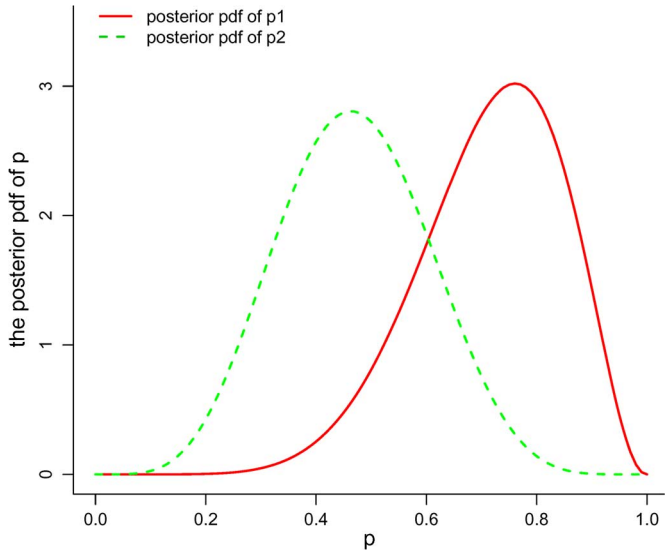

Fig. 1. The posterior pdf of p_1, and p_2.

C. Numeric Example

Assume $n = 40, r = 20, \theta_1 = 1.2, \tau_1 = 5, \theta_2 = 1.6, \tau_2 = 6$; $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1, a_1 = a_2 = 20, b_1 = b_2 = 15$, $c_1 = c_2 = 0.1$, and $d_1 = d_2 = 0.1$. For each system, we simulate 40 failure times from $f_1(t|\theta_1, \tau_1)$, and $f_2(t|\theta_2, \tau_2)$ each. Then, we get the minimum of the two failure times as the system's lifetime together with a label, the cause of the system's failure. If the cause

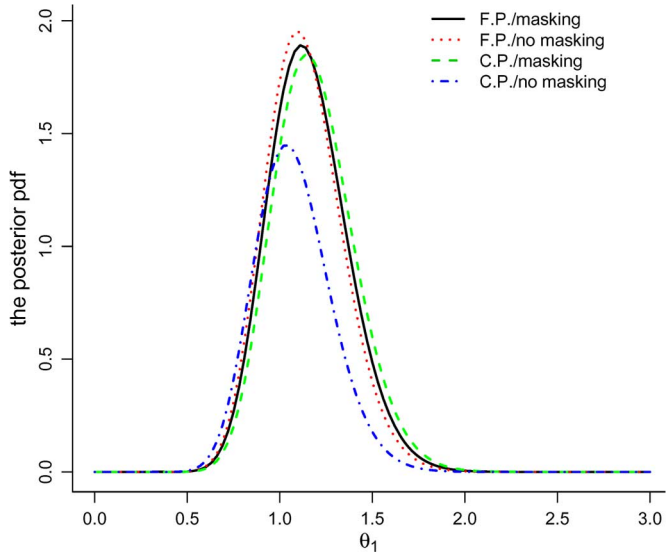

Fig. 2. The posterior pdf of θ_1.

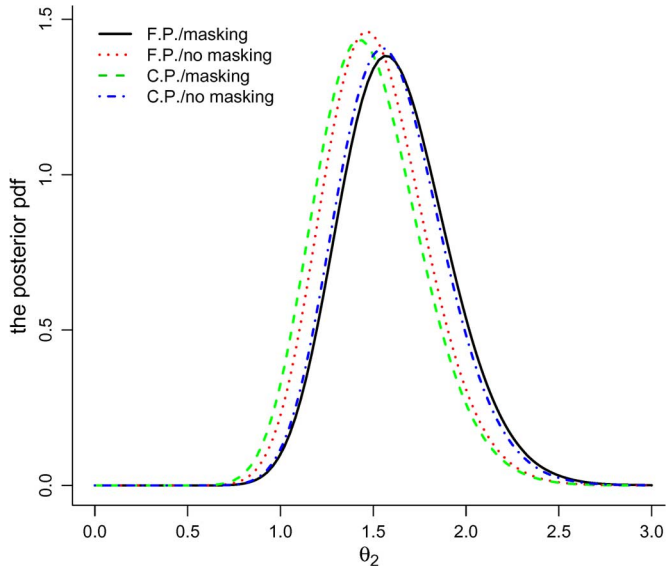

Fig. 3. The posterior pdf of θ_2.

is from the first (second) component, we randomly mask 30% (50%) of the observations. That is, the unmasked probabilities are $p_1 = 0.7$, and $p_2 = 0.5$. Data are listed in Table I

Based on the above algorithm, we found 3,000 iterations of the Gibbs sampling are enough to achieve convergence of all the posterior quantities concerned. Bayesian estimates with both flat, and conjugate priors are computed in the cases of masking, and no masking; the results are listed in Table II. The posterior means, and lengths of 90% confidence intervals for the estimation of $R(t)$, and $h(t)$ are given in Tables III and IV. Figs. 1–7 are the posterior pdf of the model parameters, reliability, and hazard function.

Some features from Tables II–IV are listed below.

1) Accuracy: the Gibbs sampling behaves quite well for the PFM. This can be judged by the posterior estimates of parameters in Table II. Each estimate is quite close to the true value.

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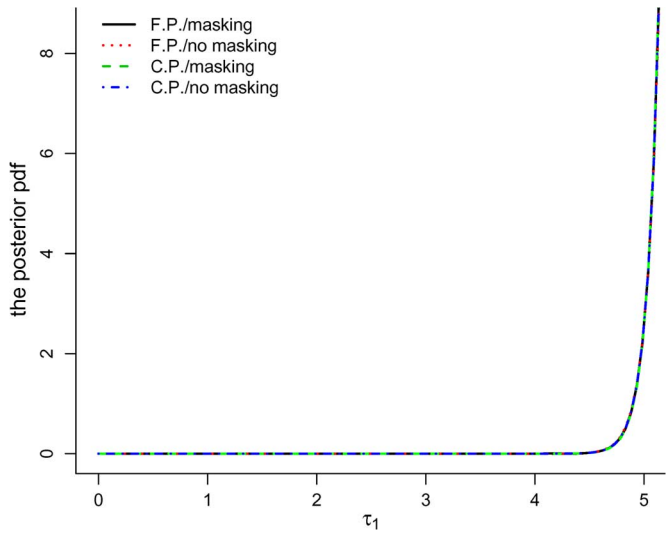

Fig. 4. The posterior pdf of τ_1.

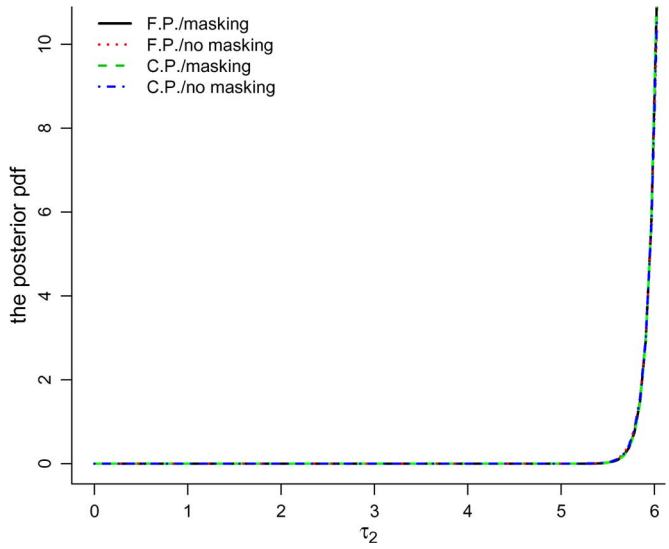

Fig. 5. The posterior pdf of τ_2.

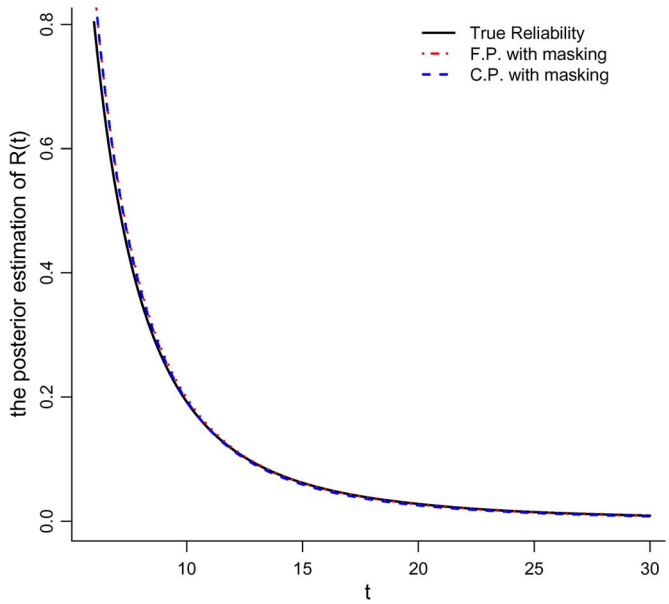

Fig. 6. The posterior estimation of the reliability function $R(t)$.

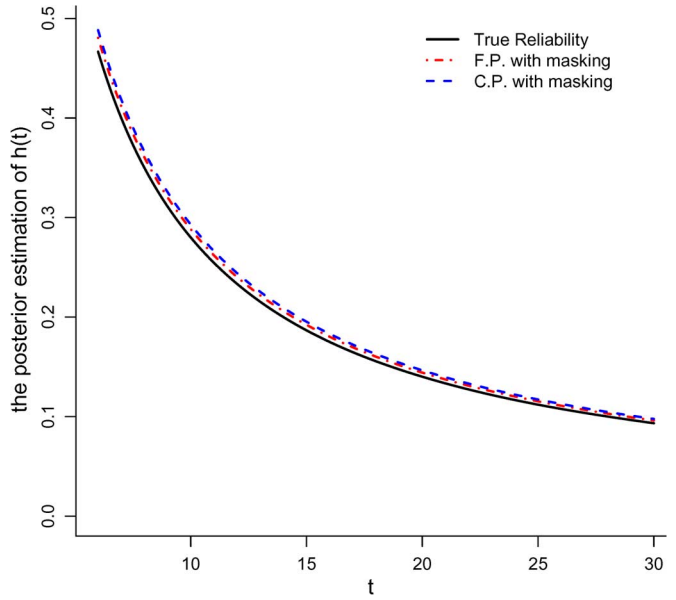

Fig. 7. The posterior estimation of the hazard function $h(t)$.

2) Difference between masking and no masking: the posterior estimate without masking is better than that with masking. For example, although the differences among lengths of 90% confidence intervals are not very large in Table III, the estimates without masking are much closer to the true value. This result happens because the masked data conceal the true failure cause, which results in the loss of information. In our Bayesian analysis, we introduce the auxiliary variable to uncover the failure cause of the system. However, that is just an interim step to help Bayesian analysis, while the true cause is unavailable.
3) Difference between flat, and conjugate priors: the posterior estimates with different priors are close to each other regardless of masking or not. This happens in part because of the choice of the values of hyperparameters (the parameters of the prior). Comparing (10) with (11), the difference is dominated by the role of hyperparameters. In the simulation, we only take small values of the hyperparameters, which account for the little difference in posterior estimation. Fig. 8 shows the effect of different values of hyperparameters on the posterior pdf of θ_1. For the determination of hyperparameters, see Berger & Sun [4].

To avoid the effect of the randomness, we repeat the above simulation 2000 times, then compute the posterior mean, and average lengths of 90% confidence intervals for $R(t)$, which are presented in Table V. Our method is satisfactory based on the MSE given in the parentheses in Table V.

IV. Conclusion, and Discussion

This paper proposes a Bayesian method of the PFM with dependent masked data. The novelty of the presented model is that the masking probabilities are assumed to be unequal. The Gibbs sampling method provides a more flexible method by introducing auxiliary variables. The simulation shows the presented method works well. It is quite general, and may therefore be easily applied to other models. Some extensions below can be further discussed.

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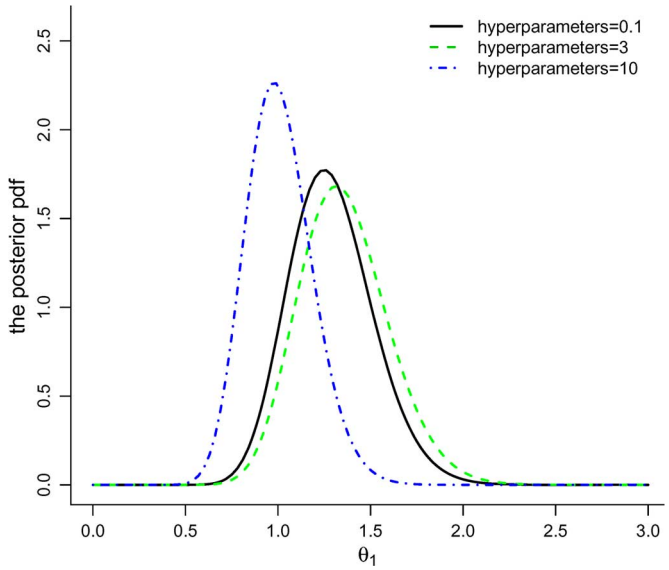

Fig. 8. The posterior pdf of θ_1 for different values of hyperparameters.

TABLE V
POSTERIOR MEANS, AND AVERAGE LENGTHS (A.L.) OF 90% CONFIDENCE INTERVALS FOR $R(t)$

True R(t)	$\hat{R}(t)$	A.L.	$\hat{R}_{no}(t)$	A.L.
.5218	.5354(1.78e-03)*	.1499	.5126(1.04e-03)	.1495
.3590	.3609(1.09e-03)	.1538	.3522(8.66e-04)	.1532
.2581	.2556(7.91e-04)	.1442	.2536(6.95e-04)	.1439
.1922	.1882(5.97e-04)	.1298	.1894(5.43e-04)	.1301
.0617	.0592(1.63e-04)	.0690	.0627(1.56e-04)	.0715
.0276	.0265(5.53e-05)	.0396	.0290(5.44e-05)	.0423
True R(t)	$\tilde{R}(t)$	A.L.	$\tilde{R}_{no}(t)$	A.L.
.5218	.5296(1.82e-03)	.1503	.5168(1.07e-03)	.1500
.3590	.3540(1.11e-03)	.1535	.3456(8.88e-04)	.1530
.2581	.2489(8.04e-04)	.1431	.2473(7.08e-04)	.1428
.1922	.1821(6.03e-04)	.1280	.1836(5.49e-04)	.1284
.0617	.0559(1.59e-04)	.0665	.0595(1.52e-04)	.0692
.0276	.0247(5.22e-05)	.0375	.0272(5.17e-05)	.0403

- This paper assumes that the series system has two components. Generally, we can extend it to a multi-component system.
- The distributions of the components in the series system can be mutually different, say Weibull, and lognormal.
- Other censored schemes, such as interval censoring, can be considered.

APPENDIX A

Based on the assumptions in Section II, we know $P(s_i = \{1,2\}|K_i = 1, t_i) = 1 - p_1$,$P(s_i = \{1,2\}|K_i = 2, t_i) = 1 - p_2$. $P(K_i = 1|t_i) = f_1(t_i)R_2(t_i)$. Using Bayesian law, we have

$$\begin{aligned} &P\left(K_i = 1|s_i = \{1,2\}, t_i\right) \\ &= \frac{P\left(s_i = \{1,2\}|K_i = 1, t_i\right) P(K_i = 1|t_i)}{\sum_{j=1}^{2} P\left(s_i = \{1,2\}|K_i = j, t_i\right) P(K_i = j|t_i)} \\ &= \frac{(1-p_1) f_1(t_i) R_2(t_i)}{(1-p_1) f_1(t_i) R_2(t_i) + (1-p_2) f_2(t_i) R_1(t_i)} \\ &= \frac{(1-p_1) h_1(t_i)}{(1-p_1) h_1(t_i) + (1-p_2) h_2(t_i)} \\ &= \frac{(1-p_1)\frac{\theta_1}{t_i}}{(1-p_1)\frac{\theta_1}{t_i} + (1-p_2)\frac{\theta_2}{t_i}} \\ &= \frac{(1-p_1)\theta_1}{(1-p_1)\theta_1 + (1-p_2)\theta_2}, \end{aligned} \tag{12}$$

which conclude the proof of Theorem 1.

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